Microeconomics III

Nash equilibrium II
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Randomization

Recall that a strategic game is a triple $\langle N, (A_i), (\succsim_i)\rangle$ where

- N is a finite set of players, and for each player $i \in N$
- a non-empty set A_i of actions
- a preference relation $\succsim_i$on the set $A = \times_{j \in N} A_j$ of possible outcomes.

or a triple $\langle N, (A_i), (u_i)\rangle$ when $\succsim_i$ can be represented by a utility function $u_i : A \rightarrow \mathbb{R}$.

Suppose that,

– each player i can randomize among all her strategies so choices are not deterministic, and

– player i's preferences over lotteries on A can be represented by vNM expected utility function.

Then, we need to add theses specifications to the primitives of the model of strategic game $\langle N, (A_i), (\succsim_i) \rangle$.

A mixed strategy of player i is $\alpha_i \in \Delta(A_i)$ where $\Delta(A_i)$ is the set of all probability distributions over A_i.

- A profile $(\alpha_i)_{i \in N}$ of mixed strategies induces a probability distribution over the set A.

- Assuming independence, the probability of an action profile (outcome) a is then

$$\prod_{i \in N} \alpha_i(a_i).$$

A vNM utility function

$$U_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R}$$

represents player i's preferences over the set of lotteries over A.

The mixed extension of a the strategic game $\langle N, (A_i), (u_i) \rangle$ is the strategic game

$$\langle N, (\Delta(A_i)), (U_i) \rangle \, .$$

Preferences toward risk

The standard model of decisions under risk (known probabilities) is based on von Neumann and Morgenstern Expected Utility Theory.

Consider a set of *lotteries*, or gambles, (outcomes and probabilities). A fundamental axiom about preferences toward risk is *independence*:

For any lotteries x, y, z and $0 < \alpha < 1$

$$x \succ y \text{ implies } \alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z.$$

Expected Utility Theory has some very convenient properties for analyzing choice under uncertainty.

To clarify, we will consider the *utility* that a consumer gets from her or his income.

More precisely, from the consumption bundle that the consumer's income can buy.

Behavioral economics

Allais (1953) I

- Choose between the two gambles:

$$A := \begin{array}{l} \overset{.33}{\nearrow}\ \$25,000 \\ \xrightarrow{.66}\ \$24,000 \\ \underset{.01}{\searrow}\ \$0 \end{array} \qquad B := \xrightarrow{1} \$24,000$$

Allais (1953) II

– Choose between the two gambles:

$$C := \begin{array}{l} \overset{.33}{\nearrow} \quad \$25,000 \\ \\ \underset{.67}{\searrow} \quad \$0 \end{array} \qquad D := \begin{array}{l} \overset{.34}{\nearrow} \quad \$24,000 \\ \\ \underset{.66}{\searrow} \quad \$0 \end{array}$$

Two results on mixed strategy Nash equilibrium

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game and $G' = \langle N, (\Delta(A_i)), (U_i) \rangle$ be its mixed extension.

[1] If $a \in NE(G)$ then $a \in NE(G')$.

[2] $\alpha \in NE(G')$ if and only if

$$U_i(\alpha_{-i}, a_i) \geq U_i(\alpha_{-i}, a_i')$$

for all a_i' and all $\alpha_i(a_i) > 0$.

[1] Proof: If $a \in NE(G)$ then

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \ \forall i \in N \text{ and } \forall a_i' \in A_i.$$

Then, by the linearity of U_i in α_i

$$U_i(a_{-i}, a_i) \geq U_i(a_{-i}, \alpha_i) \ \forall i \in N \text{ and } \forall \alpha_i \in \Delta(A_i)$$

and thus $a \in NE(G')$.

[2] Proof: Let $\alpha \in NE(G')$

Suppose that $\exists a_i \in A_i$ such that $\alpha_i(a_i) > 0$ and

$$U_i(\alpha_{-i}, a_i') \geq U_i(\alpha_{-i}, a_i) \text{ for some } a_i' \neq a_i.$$

Then, player i can increase her payoff by transferring probability from a_i to a_i' so α is not a NE.

This implies that $U_i(\alpha_{-i}, a_i) = U_i(\alpha_{-i}, a_i')$ for all a_i, a_i' in the support of α.

Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^*.

Evolutionary stable strategy (ESS)

Consider a payoff symmetric game $G = \langle \{1,2\}, (A, A), (u_i) \rangle$ where $u_1(a) = u_2(a')$ when a' is obtained from a by exchanging a_1 and a_2.

$a^* \in A$ is ESS iff for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied iff for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on ESS

[1] If a^* is an ESS then (a^*, a^*) is a NE.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an ESS.

[2] If (a^*, a^*) is a strict NE $(u(a^*, a^*) > u(a, a^*)$ for all $a \in A)$ then a^* is an ESS.

Suppose a^* is not an ESS. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a NE but not a strict NE.

[3] A 2×2 game $G = \langle \{1,2\}, (A,A), (u_i) \rangle$ where $u_i(a) \neq u_i(a')$ for any a, a' has a mixed strategy which is ESS

	a	a'
a	w, w	x, y
a'	y, x	z, z

If $w > y$ or $z > x$ then (a,a) or (a',a') are strict NE, and thus a or a' are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE (α^*, α^*) where

$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.

Strictly competitive games

A strategic game $\langle \{1,2\}, (A_i), (\succsim_i) \rangle$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

	L	R
T	$A, -A$	$B, -B$
B	$C, -C$	$D, -D$

Maxminimization

A $\max\min$ mixed strategy of player i is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

A player's payoff in $\alpha^* \in NE(G)$ is at least her $\max\min$ payoff:

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

and thus

$$U_i(\alpha^*) \geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

since the above holds for all $\alpha_i \in \triangle(A_i)$.

Two min-max results

[1] $$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

For every α'

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i})$$

and thus

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i})$$

However, since the above holds for every α'_i and α'_{-i} it must hold for the "best" and "worst" such choices

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}).$$

[2] In a zero-sum game

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)$$

$\Leftarrow$ Since $\alpha^* \in NE(G)$

$$U_1(\alpha^*) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^*) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

and since $U_1 = -U_2$ at the same time

$$U_1(\alpha^*) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^*, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)$$

Hence,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

which together with [1] gives the desired conclusion.

$\Rightarrow$ Let $\alpha_1^{\max}$ be player 1's $\max\min$ strategy and $\alpha_2^{\min}$ be player 2's $\min\max$ strategy. Then,

$$\begin{aligned}\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\max}, \alpha_2) \\ &\leq U_1(\alpha_1^{\max}, \alpha_2) \ \forall \alpha_2 \in \Delta A_2\end{aligned}$$

and

$$\begin{aligned}\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\min}) \\ &\geq U_1(\alpha_1, \alpha_2^{\min}) \ \forall \alpha_1 \in \Delta A_1\end{aligned}$$

But

$$\begin{aligned}\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \\ &= U_1(\alpha_1^{\max}, \alpha_2^{\min})\end{aligned}$$

implies that

$$U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)$$

$\forall \alpha_2 \in \Delta A_2$ and $\forall \alpha_1 \in \Delta A_1$.

Hence, $(\alpha_1^{\max}, \alpha_2^{\min})$ is an equilibrium.

Interchangeability

If α and α' are NE in a zero-sum game, then so are (α_1, α'_2) and (α'_1, α_2).

– Since α and α' are equilibria

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \text{ and } U_2(\alpha'_1, \alpha'_2) \geq U_2(\alpha'_1, \alpha_2),$$

and because $U_1 = -U_2$

$$U_1(\alpha'_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha_2).$$

Therefore,

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \geq U_1(\alpha'_1, \alpha'_2). \tag{1}$$

and similar analysis gives that

$$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2). \tag{2}$$

– (1) and (2) yield

$$U_1(\alpha_1, \alpha_2) = U_1(\alpha_1', \alpha_2) = U_1(\alpha_1, \alpha_2') = U_1(\alpha_1', \alpha_2')$$

– Since α is an equilibrium

$$U_2(\alpha_1, \alpha_2'') \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha_2')$$

for any $\alpha_2'' \in \Delta A_2$, and since α' is an equilibrium

$$U_1(\alpha_1'', \alpha_2') \leq U_1(\alpha_1', \alpha_2') = U_1(\alpha_1, \alpha_2')$$

for any $\alpha_1'' \in \Delta A_1$. Therefore, (α_1, α_2') is an equilibrium and similarly also (α_1, α_2').