Microeconomics III

Subgame perfect equilibrium
(Apr 29, 2012)

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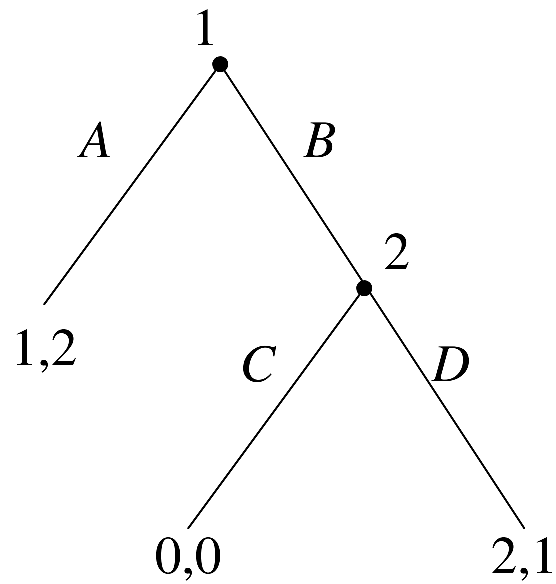
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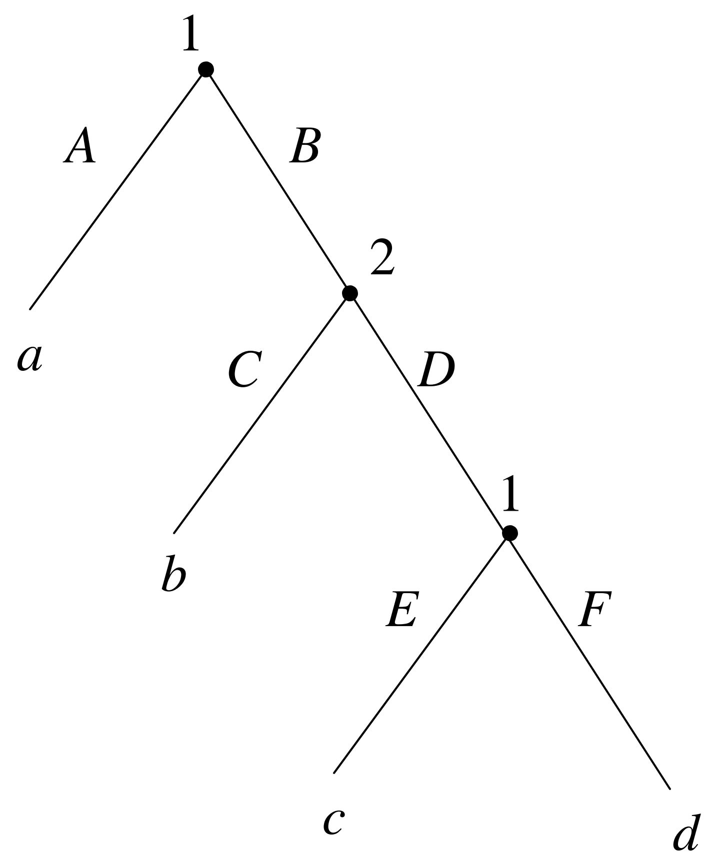
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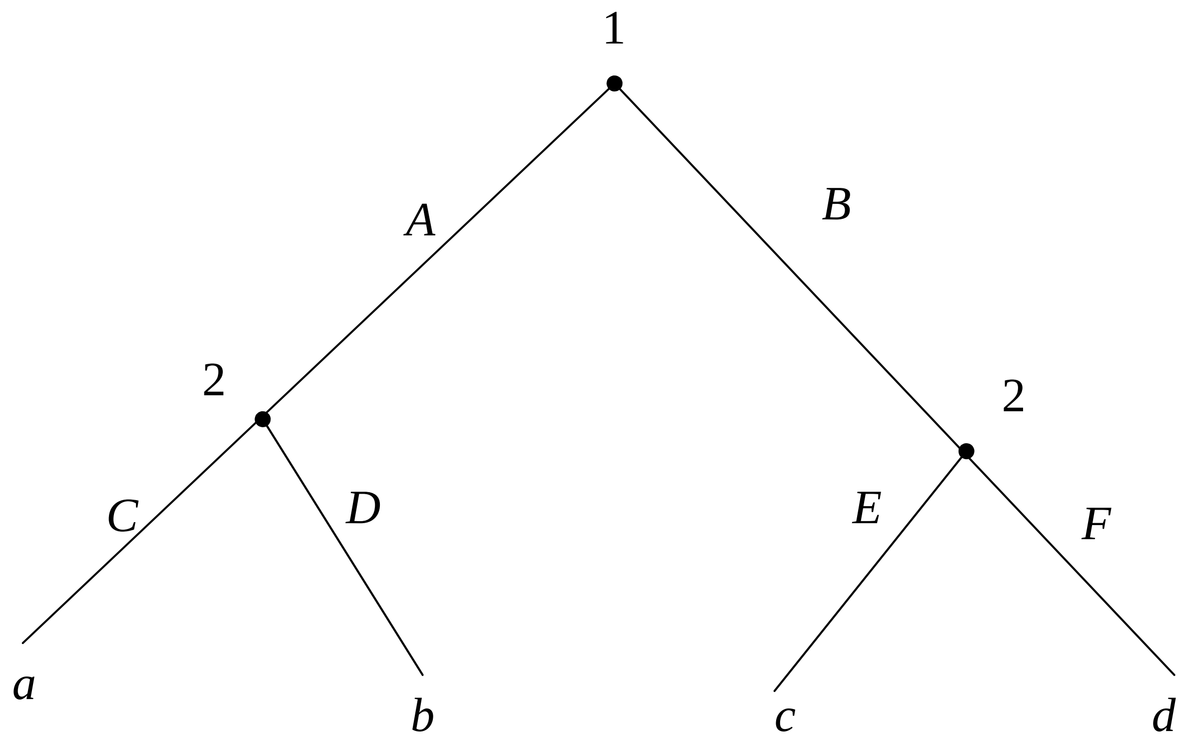
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Perfect information

A finite extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ consists of

- A set N of players.
- A set H of sequences (histories) where $\varnothing \in H$ and for any $L < K$
$$(a^k)_{k=1}^K \in H \Longrightarrow (a^k)_{k=1}^L \in H.$$
- A player function $P : H \backslash Z \to N$ where $h \in Z \subseteq H$ if $(h, a) \notin H$.
- A preference relation $\succsim_i$ on Z for each player $i \in N$.

Strategies, outcomes and Nash equilibrium

A strategy

$$s_i : h \to A(h) \text{ for every } h \in H \backslash Z \text{ such that } P(h) = i.$$

A Nash equilibrium of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile $(s_i^*)_{i \in N}$ such that for any $i \in N$

$$O(s^*) \succsim_i O(s_i, s_{-i}^*) \ \forall s_i$$

where $O(s) = (a^1, ..., a^K) \in Z$ such that

$$s_{P(a^1,...,a^k)}(a^1, ..., a^k) = a^{k+1}$$

for any $0 \leq k < K$ (an outcome).

The (reduced) strategic form

$G = \left\langle N, (S_i), (\succsim_i') \right\rangle$ is the strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S_i is player i's strategy set in Γ and $\succsim_i'$ is defined by

$$s \succsim_i' s' \Leftrightarrow O(s) \succsim_i' O(s') \; \forall s, s' \in \times_{i \in N} S_i$$

$G = \left\langle N, (S_i'), (\succsim_i'') \right\rangle$ is the reduced strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S_i' contains one member of *equivalent* strategies in S_i, that is,

$$s_i, s_i' \in S_i \text{ are equivalent if } (s_i, s_{-i}) \sim_j' (s_i', s_{-i}) \forall j \in N,$$

and $\succsim_i''$defined over $\times_{j \in N} S_j'$ and induced by $\succsim_i'$.

Subgames and subgame perfection

A <u>subgame</u> of Γ that follows the history h is the game $\Gamma(h)$

$$\langle N, H|_h, P|_h, (\succsim_i|_h)\rangle$$

where for each $h' \in H_h$

$(h, h') \in H, P|_h(h') = P(h, h')$ and $h' \succsim_i|_h h'' \Leftrightarrow (h, h') \succsim_i (h, h'')$.

$s^* \in \times_{i \in N} S_i$ is a <u>subgame perfect equilibrium</u> (SPE) of Γ if

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succsim_i|_h O_h(s_i|_h, s_{-i}^*|_h)$$

for each $i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$ and for any $s_i|_h$.

Thus, the equilibrium of the full game must induce on equilibrium on every subgame.

Backward induction and Kuhn's theorems

Let Γ be a finite extensive game with perfect information

- Γ has a SPE (Kuhn's theorem).

  The proof is by backward induction (Zermelo, 1912) which is also an algorithm for calculating the set of SPE.

- Γ has a unique SPE if there is no $i \in N$ such that $z \sim_i z'$ for any $z, z' \in Z$.

- Γ is dominance solvable if $z \sim_i z' \;\; \exists i \in N$ then $z \sim_j z' \; \forall j \in N$ (but elimination of weakly dominated strategies in G may eliminate the SPE in Γ).

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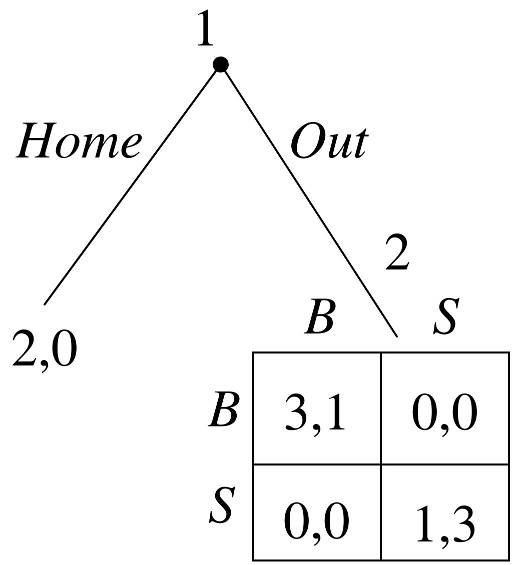
Forward induction

- Backward induction cannot always ensure a self-enforcing equilibrium (forward and backward induction).

- In an extensive game with simultaneous moves, players interpret a deviation as a signal about future play.

- The concept of iterated <u>weak</u> dominance can be used to capture forward and backward induction.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