Microeconomics III

Bargaining II
The axiomatic approach
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Nash (1953) bargaining

A bargaining situation is a tuple $\langle N, A, D, (\succsim_i)\rangle$ where

- N is a set of players or bargainers ($N = \{1, 2\}$),
- A is a set of agreements/outcomes,
- D is a disagreement outcome, and
- $\succsim_i$ is a preference ordering over the set of lotteries over $A \cup \{D\}$.

The objects N, A, D and $\succsim_i$ for $i = \{1, 2\}$ define a bargaining situation.

$\succsim_1$ and $\succsim_2$ satisfy the assumption of vNM so for each i there is a utility function $u_i : A \cup \{D\} \to \mathbb{R}$.

$\langle S, d \rangle$ is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$ for $a \in A$ the set of all utility pairs, and
- $d = (u_1(D), u_2(D))$.

A <u>bargaining problem</u> is a pair $\langle S, d \rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$ and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems $\langle S, d \rangle$ is denoted by B.

A <u>bargaining solution</u> is a function $f : B \to \mathbb{R}^2$ such that f assigns to each bargaining problem $\langle S, d \rangle \in B$ a unique element in S.

Nash's axioms

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

Invariance to equivalent utility representations (INV)

$\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ if

$$d'_i = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if $\alpha_i > 0$ for $i = 1, 2$ then $\langle S', d' \rangle$ is itself a bargaining problem.

If $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ where $\alpha_i > 0$ for each i, then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for $i = 1, 2$. Hence, $\langle S', d' \rangle$ and $\langle S, d \rangle$ represent the same situation.

INV requires that the utility outcome of the bargaining problem co-vary with representation of preferences.

The physical outcome predicted by the bargaining solution is the same for $\langle S', d' \rangle$ and $\langle S, d \rangle$.

A corollary of INV is that we can restrict attention to $\langle S, d \rangle$ such that

$S \subset \mathbb{R}^2_+$,

$S \cap \mathbb{R}^2_{++} \neq \emptyset$, and

$d = (0, 0) \in S$ (reservation utilities).

Symmetry (SYM)

A bargaining problem $\langle S, d \rangle$ is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$. If the bargaining problem $\langle S, d \rangle$ is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by $\langle S, d \rangle$.

Hence, if players are the same the bargaining solution must assign the same utility to each player.

Independence of irrelevant alternatives (IIA)

If $\langle S, d \rangle$ and $\langle T, d \rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$ then

$$f(S, d) = f(T, d)$$

If T is available and players agree on $s \in S \subset T$ then they agree on the same s if only S is available.

IIA excludes situations in which the fact that a certain agreement is available influences the outcome.

Weak Pareto efficiency (WPO)

If $\langle S, d\rangle$ is a bargaining problem where $s \in S$ and $t \in S$, and $t_i > s_i$ for $i = 1, 2$ then $f(S, d) \neq s$.

In words, players never agree on an outcome s when there is an outcome t in which both are better off.

Hence, players never disagree since by assumption there is an outcome s such that $s_i > d_i$ for each i.

$\underline{SYM \text{ and } WPO}$

restrict the solution on single bargaining problems.

$\underline{INV \text{ and } IIA}$

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by $f^N(S, d)$, satisfying SYM, WPO, INV and IIA.

Nash's solution

The unique bargaining solution $f^N : B \to \mathbb{R}^2$ satisfying SYM, WPO, INV and IIA is given by

$$f^N(S,d) = \underset{(d_1,d_2)\le(s_1,s_2)\in S}{\arg\max} (s_1 - d_1)(s_2 - d_2)$$

and since we normalize $(d_1, d_2) = (0, 0)$

$$f^N(S,0) = \underset{(s_1,s_2)\in S}{\arg\max} s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

Proof

Pick a compact and convex set $S \subset \mathbb{R}^2_+$ where $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Step 1: f^N is well defined.

– Existence: the set S is compact and the function $f = s_1 s_2$ is continuous.

– Uniqueness: f is strictly quasi-conacave on S and the set S is convex.

Step 2: f^N is the only solution that satisfies SYM, WPO, INV and IIA.

Suppose there is another solution f that satisfies SYM, WPO, INV and IIA.

Let

$$S' = \{(\frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)}) : (s_1, s_2) \in S\}$$

and note that $s'_1 s'_2 \leq 1$ for any $s' \in S'$, and thus $f^N(S', 0) = (1, 1)$.

Since S' is bounded we can construct a set T that is symmetric about the 45° line and contains S'

$$T = \{(a, b) : a + b \leq 2\}$$

By WPO and SYM we have $f(T, 0) = (1, 1)$, and by IIA we have $f(S', 0) = f(T, 0) = (1, 1)$.

By INV we have that $f(S', 0) = f^N(S', 0)$ if and only if $f(S, 0) = f^N(S, 0)$ which completes the proof.

Is any axiom superfluous?

$\underline{INV}$

The bargaining solution given by the maximizer of

$$g(s_1, s_2) = \sqrt{s_1} + \sqrt{s_2}$$

over $\langle S, 0 \rangle$ where $S := co\{(0,0),(1,0),(0,2)\}$.

This solution satisfies WPO, SYM and IIA (maximizer of an increasing function). The maximizer of g for this problem is $(1/3, 4/3)$ while $f^N = (1/2, 1)$.

$\underline{SYM}$

The family of solutions $\{f^\alpha\}_{\alpha\in(0,1)}$ over $\langle S, 0\rangle$ where

$$f^\alpha(S,d) = \operatorname*{arg\,max}_{(d_1,d_2)\leq(s_1,s_2)\in S} (s_1 - d_1)^\alpha (s_2 - d_2)^{1-\alpha}$$

is called the asymmetric Nash solution.

Any f^α satisfies INV, IIA and WPO by the same arguments used for f^N.

For $\langle S, 0\rangle$ where $S := co\{(0,0),(1,0),(0,1)\}$ we have $f^\alpha(S,0) = (\alpha, 1-\alpha)$ which is different from f^N for any $\alpha \neq 1/2$.

$\underline{WPO}$

Consider the solution f^d given by $f^d(S, d) = d$ which is different from f^N. f^d satisfies INV, SYM and IIA.

WPO in the Nash solution can be replaced with strict individual rationality (SIR) $f(S, d) >> d$.