UC Berkeley
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Berkeley MBA for Executives Program

Game Theory
(XMBA 296)

Block 2
Evolutionary stability, auctions, and bargaining (axiomatic approach)
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Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^*.

Evolutionary stable strategy (ESS)

Consider a two-player payoff symmetric game

$$G = \langle \{1, 2\}, (A, A), (u_1, u_2) \rangle$$

where

$$u_1(a_1, a_2) = u_2(a_2, a_1)$$

(players exchanging a_1 and a_2).

$a^* \in A$ is ESS if and only if for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if and only if for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on ESS

[1] If a^* is an ESS then (a^*, a^*) is a NE.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an ESS.

[2] If (a^*, a^*) is a strict NE $(u(a^*, a^*) > u(a, a^*)$ for all $a \in A)$ then a^* is an ESS.

Suppose a^* is not an ESS. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a NE but not a strict NE.

[3] The two-player two-action game

	a	a'
a	w, w	x, y
a'	y, x	z, z

has a strategy which is ESS.

If $w > y$ or $z > x$ then (a, a) or (a', a') are strict NE, and thus a or a' are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE (α^*, α^*) where

$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.

Auctions

From Babylonia to eBay, auctioning has a very long history.

- Babylon:
  - women at marriageable age.
- Athens, Rome, and medieval Europe:
  - rights to collect taxes,
  - dispose of confiscated property,
  - lease of land and mines,

  and more...

- Auctions, broadly defined, are used to allocate significant economics resources.

  Examples: works of art, government bonds, offshore tracts for oil exploration, radio spectrum, and more.

- Auctions take many forms. A game-theoretic framework enables to understand the consequences of various auction designs.

- Game theory can suggest the design likely to be most effective, and the one likely to raise the most revenues.

Types of auctions

Sequential / simultaneous

Bids may be called out sequentially or may be submitted simultaneously in sealed envelopes:

- English (or oral) – the seller actively solicits progressively higher bids and the item is soled to the highest bidder.

- Dutch – the seller begins by offering units at a "high" price and reduces it until all units are soled.

- Sealed-bid – all bids are made simultaneously, and the item is sold to the highest bidder.

First-price / second-price

The price paid may be the highest bid or some other price:

- First-price – the bidder who submits the highest bid wins and pay a price equal to her bid.

- Second-prices – the bidder who submits the highest bid wins and pay a price equal to the second highest bid.

Variants: all-pay (lobbying), discriminatory, uniform, Vickrey (William Vickrey, Nobel Laureate 1996), and more.

Private-value / common-value

Bidders can be certain or uncertain about each other's valuation:

- In private-value auctions, valuations differ among bidders, and each bidder is certain of her own valuation and can be certain or uncertain of every other bidder's valuation.

- In common-value auctions, all bidders have the same valuation, but bidders do not know this value precisely and their estimates of it vary.

First-price auction (with perfect information)

To define the game precisely, denote by v_i the value that bidder i attaches to the object. If she obtains the object at price p then her payoff is $v_i - p$.

Assume that bidders' valuations are all different and all positive. Number the bidders 1 through n in such a way that

$$v_1 > v_2 > \cdots > v_n > 0.$$

Each bidder i submits a (sealed) bid b_i. If bidder i obtains the object, she receives a payoff $v_i - b_i$. Otherwise, her payoff is zero.

Tie-breaking – if two or more bidders are in a tie for the highest bid, the winner is the bidder with the highest valuation.

In summary, a first-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.

- Actions: the set of possible bids b_i of each player i (nonnegative numbers).

- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \overline{b} & \text{if} \quad b_i = \overline{b} \text{ and } v_i > v_j \text{ if } b_j = \overline{b} \\ 0 & \text{if} \quad b_i < \overline{b} \end{cases}$$

  where $\overline{b}$ is the highest bid.

The set of Nash equilibria is the set of profiles $(b_1, ..., b_n)$ of bids with the following properties:

$$\begin{array}{ll} [1] & v_2 \leq b_1 \leq v_1 \\ [2] & b_j \leq b_1 \text{ \underline{for all} } j \neq 1 \\ [3] & b_j = b_1 \text{ \underline{for some} } j \neq 1 \end{array}$$

It is easy to verify that all these profiles are Nash equilibria. It is harder to show that there are no other equilibria. We can easily argue, however, that there is no equilibrium in which player 1 does not obtain the object.

$\Longrightarrow$ The first-price sealed-bid auction is socially efficient, but does not necessarily raise the most revenues.

Second-price auction (with perfect information)

A second-price sealed-bid auction with perfect information is the following strategic game:

– Players: the n bidders.

– Actions: the set of possible bids b_i of each player i (nonnegative numbers).

– Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \overline{b} & \text{if} \quad b_i > \overline{b} \text{ or } b_i = \overline{b} \text{ and } v_i > v_j \text{ if } b_j = \overline{b} \\ 0 & \text{if} \quad b_i < \overline{b} \end{cases}$$

where $\overline{b}$ is the highest bid submitted by a player other than i.

First note that for any player i the bid $b_i = v_i$ is a (weakly) dominant action (a "truthful" bid), in contrast to the first-price auction.

The second-price auction has many equilibria, but the equilibrium $b_i = v_i$ for all i is distinguished by the fact that every player's action dominates all other actions.

Another equilibrium in which player $j \neq 1$ obtains the good is that in which

[1] $b_1 < v_j$ and $b_j > v_1$
[2] $b_i = 0$ <u>for all</u> $i \neq \{1, j\}$

Common-value auctions and the winner's curse

Suppose we all participate in a sealed-bid auction for a jar of coins. Once you have estimated the amount of money in the jar, what are your bidding strategies in first- and second-price auctions?

The winning bidder is likely to be the bidder with the largest positive error (the largest overestimate).

In this case, the winner has fallen prey to the so-called the winner's curse. Auctions where the winner's curse is significant are oil fields, spectrum auctions, pay per click, and more.

Bargaining

Nash's (1950) work is the starting point for formal bargaining theory.

The bargaining problem consists of

- a set of utility pairs that can be derived from possible agreements, and
- a pair of utilities which is designated to be a disagreement point.

Bargaining solution

The bargaining solution is a function that assigns a unique outcome to every bargaining problem.

Nash's bargaining solution is the first solution that

- satisfies four plausible conditions, and
- has a simple functional form, which make it convenient to apply.

A bargaining situation

A bargaining situation:

- N is a set of players or bargainers,
- A is a set of agreements/outcomes,
- D is a disagreement outcome, and

$\langle S, d \rangle$ is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$ for $a \in A$ the set of all utility pairs, and $d = (u_1(D), u_2(D))$.

A bargaining problem is a pair $\langle S, d\rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$ and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems $\langle S, d\rangle$ is denoted by B.

A bargaining solution is a function $f : B \to \mathbb{R}^2$ such that f assigns to each bargaining problem $\langle S, d\rangle \in B$ a unique element in S.

Nash's axioms

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

Invariance to equivalent utility representations (INV)

$\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i' \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ if

$$d_i' = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if $\alpha_i > 0$ for $i = 1, 2$ then $\langle S', d' \rangle$ is itself a bargaining problem.

If $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ where $\alpha_i > 0$ for each i, then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for $i = 1, 2$. Hence, $\langle S', d' \rangle$ and $\langle S, d \rangle$ represent the same situation.

Symmetry (SYM)

A bargaining problem $\langle S, d \rangle$ is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$. If the bargaining problem $\langle S, d \rangle$ is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by $\langle S, d \rangle$.

Hence, if players are the same the bargaining solution must assign the same utility to each player.

Independence of irrelevant alternatives (IIA)

If $\langle S, d\rangle$ and $\langle T, d\rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$ then

$$f(S, d) = f(T, d)$$

If T is available and players agree on $s \in S \subset T$ then they agree on the same s if only S is available.

IIA excludes situations in which the fact that a certain agreement is available influences the outcome.

Weak Pareto efficiency (WPO)

If $\langle S, d \rangle$ is a bargaining problem where $s \in S$ and $t \in S$, and $t_i > s_i$ for $i = 1, 2$ then $f(S, d) \neq s$.

In words, players never agree on an outcome s when there is an outcome t in which both are better off.

Hence, players never disagree since by assumption there is an outcome s such that $s_i > d_i$ for each i.

$\underline{SYM \text{ and } WPO}$

restrict the solution on single bargaining problems.

$\underline{INV \text{ and } IIA}$

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by $f^N(S,d)$, satisfying SYM, WPO, INV and IIA.

Nash's solution

The unique bargaining solution $f^N : B \to \mathbb{R}^2$ satisfying SYM, WPO, INV and IIA is given by

$$f^N(S, 0) = \underset{(s_1,s_2)\in S}{\arg\max}\, s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

Proof

Pick a compact and convex set $S \subset \mathbb{R}^2_+$ where $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Step 1: f^N is well defined.

- Existence: the set S is compact and the function $f = s_1 s_2$ is continuous.

- Uniqueness: f is strictly quasi-conacave on S and the set S is convex.

Step 2: f^N is the only solution that satisfies SYM, WPO, INV and IIA.

Suppose there is another solution f that satisfies SYM, WPO, INV and IIA.

Let

$$S' = \{(\frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)}) : (s_1, s_2) \in S\}$$

and note that $s_1' s_2' \leq 1$ for any $s' \in S'$, and thus $f^N(S', 0) = (1, 1)$.

Since S' is bounded we can construct a set T that is symmetric about the 45° line and contains S'

$$T = \{(a, b) : a + b \leq 2\}$$

By WPO and SYM we have $f(T, 0) = (1, 1)$, and by IIA we have $f(S', 0) = f(T, 0) = (1, 1)$.

By INV we have that $f(S', 0) = f^N(S', 0)$ if and only if $f(S, 0) = f^N(S, 0)$ which completes the proof.