UC Berkeley
Haas School of Business
Game Theory
(EMBA 296 & EWMBA 211)
Summer 2016

Review, oligopoly, auctions,

and risk preferences and social preferences

Block 3
Jul 7-9, 2016

Food for thought

LUPI

Many players simultaneously chose an integer between 1 and 99,999. Whoever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

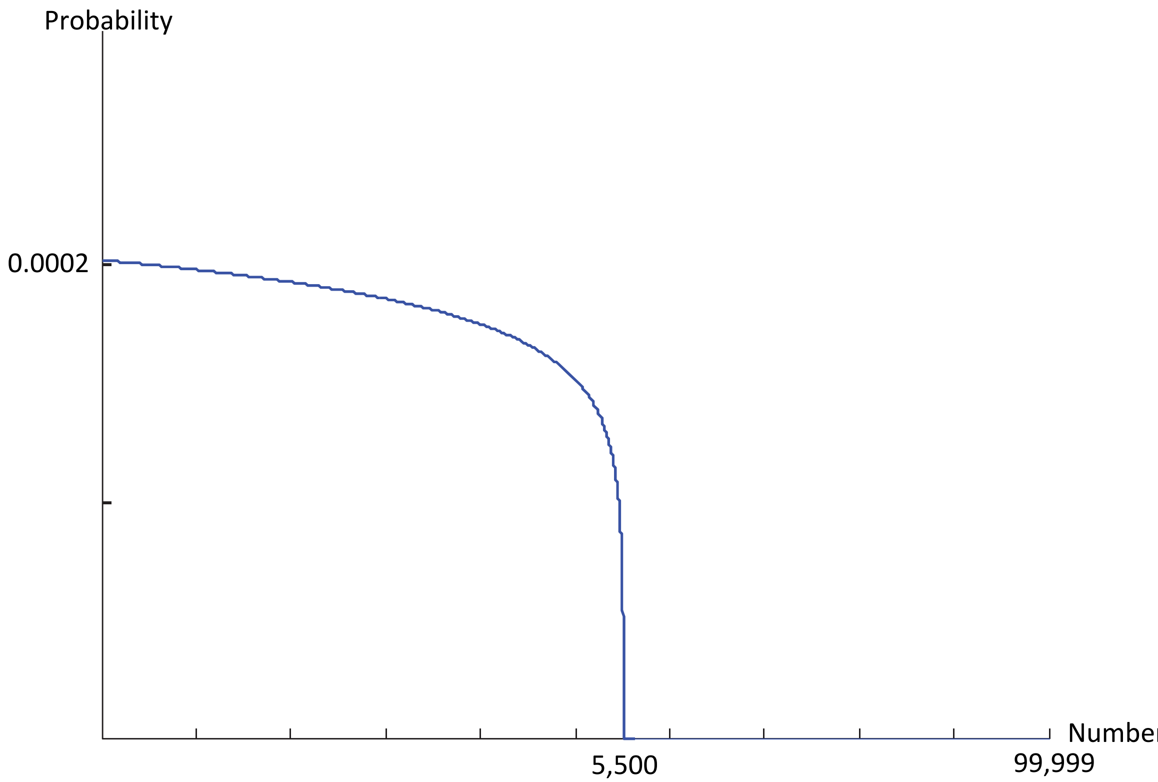
The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.

Games with population uncertainty relax the assumption that the exact number of players is common knowledge.

In particular, in a Poisson game (Myerson; 1998, 2000) the number of players N is a random variable that follows a Poisson distribution with mean n so the probability that $N = k$ is given by

$$\frac{e^{-n}n^k}{k!}$$

In the Swedish game the average number of players was $n = 53,783$ and number choices were positive integers up to $99,999$.


Probability
0.0002
Number
5,500
99,999

Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

In Morra there are two players, each of whom has four (relevant) actions, S_1G_2, S_1G_3, S_2G_3, and S_2G_4, where S_iG_j denotes the strategy (Show i, Guess j).

The payoffs in the game are as follows

	S_1G_2	S_1G_3	S_2G_3	S_2G_4
S_1G_2	$0,0$	$2,-2$	$-3,3$	$0,0$
S_1G_3	$-2,2$	$0,0$	$0,0$	$3,-3$
S_2G_3	$3,-3$	$0,0$	$0,0$	$-4,4$
S_2G_4	$0,0$	$-3,3$	$4,-4$	$0,0$

Maximal game
(sealed-bid second-price auction)

Two bidders, each of whom privately observes a signal X_i that is independent and identically distributed (i.i.d.) from a uniform distribution on $[0, 10]$.

Let $X^{\max} = \max\{X_1, X_2\}$ and assume the ex-post common value to the bidders is $X^{\max}$.

Bidders bid in a sealed-bid second-price auction where the highest bidder wins, earns the common value $X^{\max}$ and pays the second highest bid.

Homework review

1/1 **Penalty Kick**

There are two players, 1 (kicker) and 2 (goalie). Each has two actions, $a_i \in \{L, R\}$ to denote left or right.

The kicker scores when they choose opposite directions while the goalie saves if they choose the same direction so preferences ordering over outcomes is given by

$$
\begin{aligned}
(L, R) \;&\sim\; {}_1(R, L) \succ_1 (L, L) \sim_1 (R, R) \\
(L, R) \;&\sim\; {}_2(R, L) \prec_2 (L, L) \sim_2 (R, R)
\end{aligned}
$$

The game can be described as follows:

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

or equivalently

	L	R
L	$0, 0$	$1, -1$
R	$1, -1$	$0, 0$

The game has a unique mixed strategy Nash equilibrium $p = q = 1/2$.

1/2 **Meeting Up**

There are two players. Each has two actions, $a_i \in \{C, S\}$ to denote Sutro or Coit. preferences ordering over outcomes is given by

$$(C, C) \sim_1 (S, S) \succ_1 (C, S) \sim_1 (S, C)$$
$$(C, C) \sim_2 (S, S) \succ_2 (C, S) \sim_2 (S, C)$$

so the game can be described as follows:

	S	C
S	1, 1	0, 0
C	0, 0	1, 1

1/5 **Public Good Contribution**

- An indivisible public project with cost 2 and 3 players, each of whom has an endowment of 1 tokens.

- The players simultaneously make a contribution to the project, which is carried out if and only if the sum of the contributions is large enough to meet its cost.

- If the project is completed, each player receives 3 tokens *plus* to the number of tokens retained from his endowment.

The set of players is $N = \{1, 2, 3\}$ and each has a strategy set $S_i = \{0, 1\}$ where 0 denotes not contributing and 1 is contributing.

The payoffs of player i denoted by v_i from a profile of strategies (s_1, s_2, s_3) is given by

$$v_i(s_1, s_2, s_3) = \begin{cases} 4 & \text{if } s_i = 0 \text{ and } s_j = 1 \text{ for both } j \neq i \\ 3 & \text{if } s_i = 1 \text{ and } s_j = 1 \text{ for some } j \neq i \\ 1 & \text{if } s_i = 0 \text{ and } s_j = 0 \text{ for both } j \neq i \\ 0 & \text{if } s_i = 1 \text{ and } s_j = 0 \text{ for both } j \neq i \end{cases}$$

- The game has the following pure-strategy equilibria:
  - There exists a pure-strategy Nash equilibrium with no player contributes.
  - Conversely, there exist multiple pure-strategy equilibria in which exactly two players contribute.
- The game also possesses mixed-strategy equilibria in which the project is completed with positive probability.
- What happens if players simultaneously make *irreversible* contributions to the project at two dates?

1/8 **Campaigning**

	P	B	N
P	0.5, 0.5	0, 1	0.3, 0.7
B	1, 0	0.5, 0.5	0.4, 0.6
N	0.7, 0.3	0.6, 0.4	0.5, 0.5

	B	N
B	0.5, 0.5	0.4, 0.6
N	0.6, 0.4	0.5, 0.5

	N
N	0.5, 0.5

1/8 **Synergies**

Two managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest.

In particular, the payoff function for player i from effort levels (e_i, e_j) is

$$v_i(e_i, e_j) = ae_i + e_i e_j - e_i^2.$$

The best response function of player i is given by

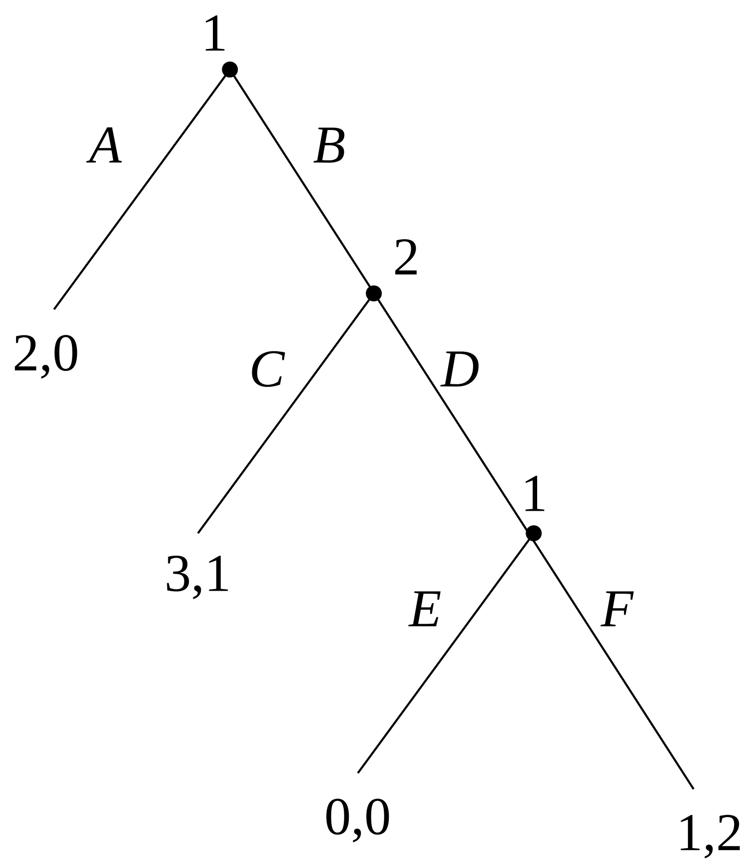
$$BR_i(e_j) = \frac{a + e_j}{2}$$

because it is the solution of the first-order condition for maximizing her payoff.

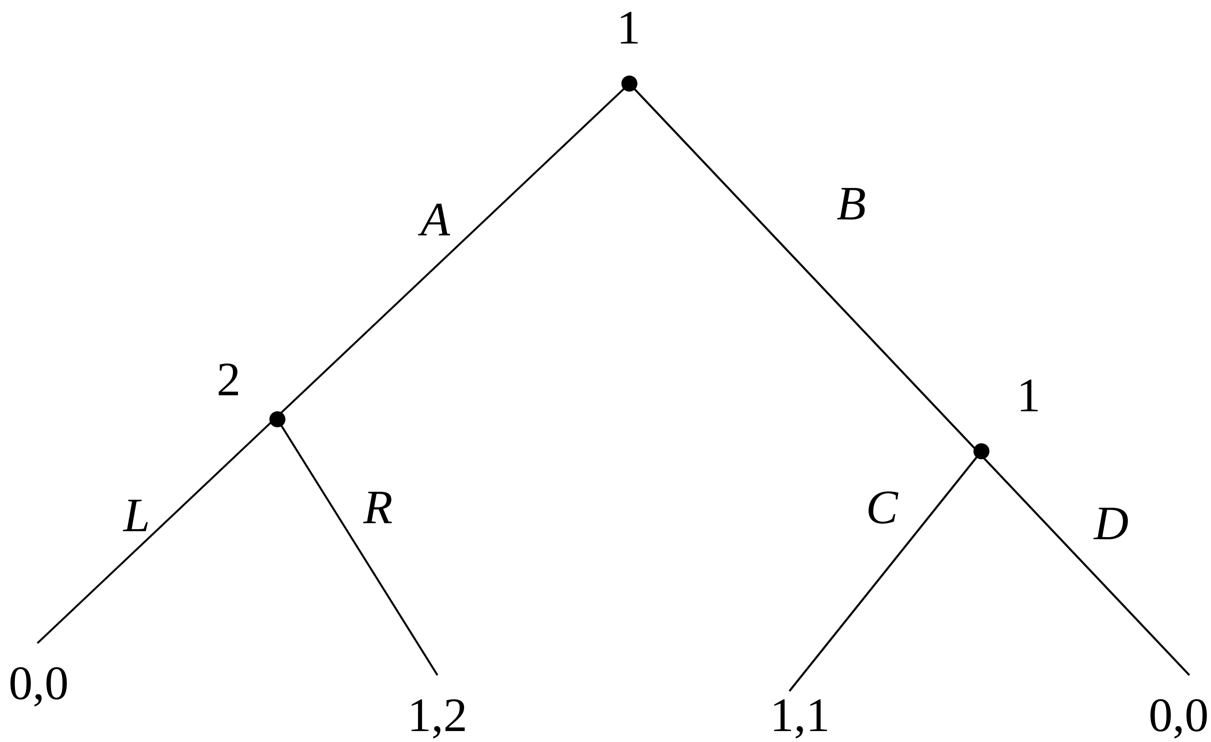
The Nash equilibrium of this game, is the solution, denoted by e_1^* and e_2^*, of

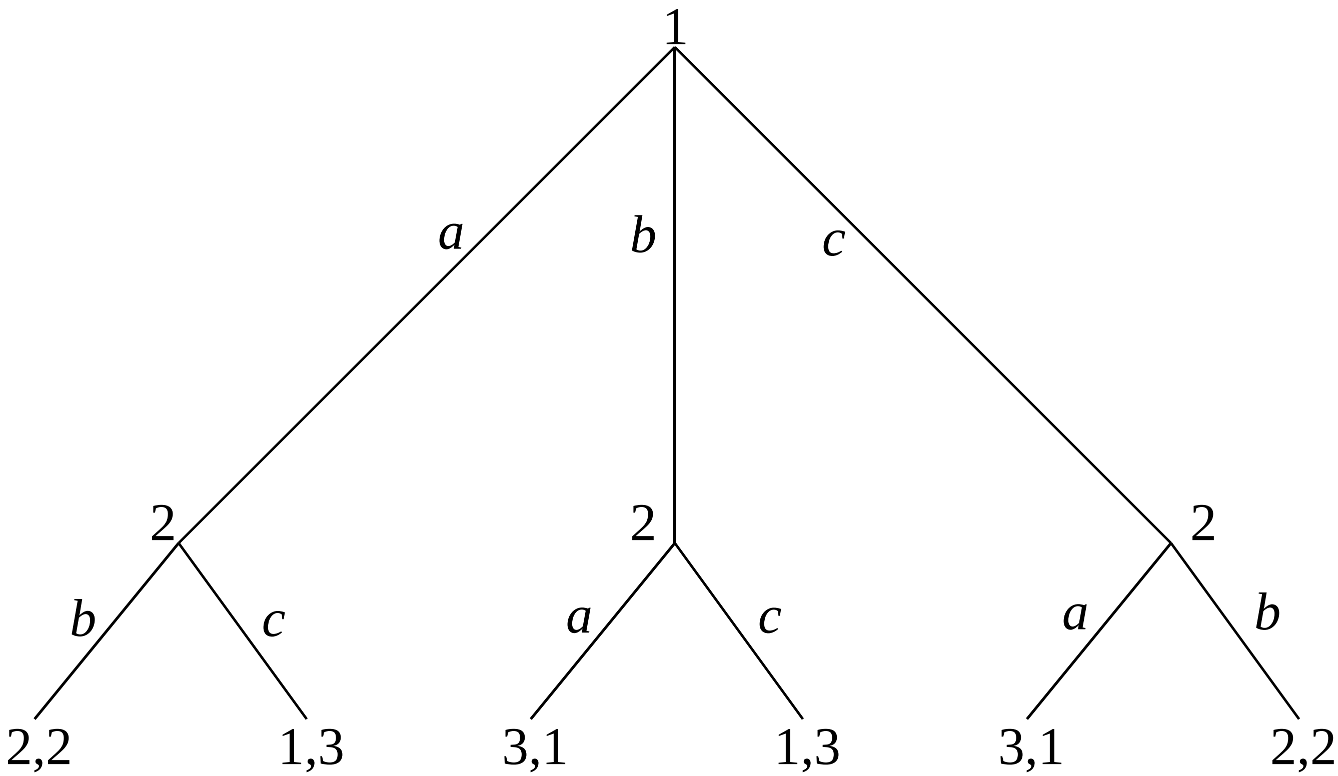
$$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2}$$

which yield $e_1^* = e_2^* = a$. Is the Nash equilibrium socially optimal?

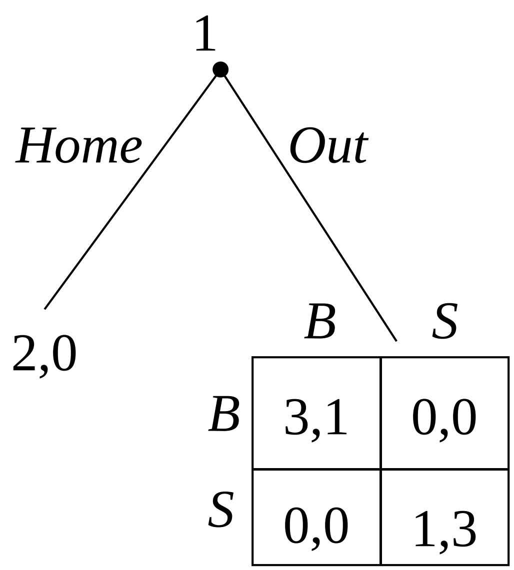

1
A
B
2,0
2
C
D
3,1
1
E
F
0,0
1,2

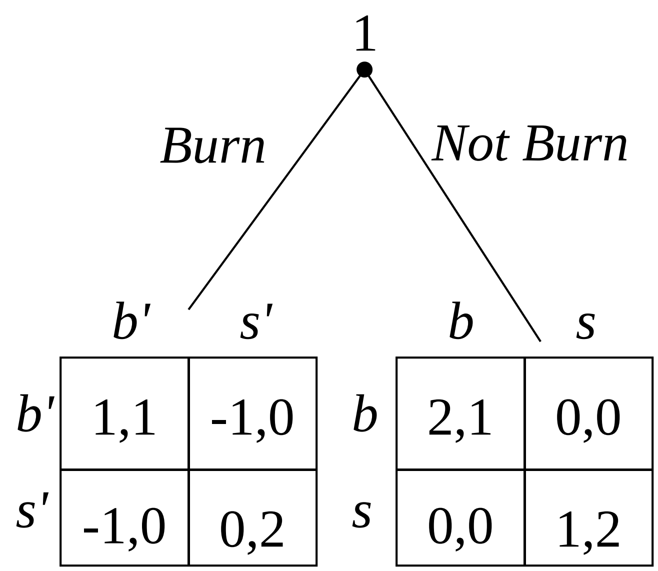
And one more example :-)




1
a
b
c
2
2
2
b
c
a
c
a
b
2,2
1,3
3,1
1,3
3,1
2,2

2/9 (two variants)




1
Burn
Not Burn
b′
s′
b′
s′
1,1
-1,0
-1,0
0,2
b
s
b
s
2,1
0,0
0,0
1,2

	$b'b$	$b's$	$s'b$	$s's$
$Bb'b$	1, 1	1, 1	−1, 0	−1, 0
$Bb's$	1, 1	1, 1	−1, 0	−1, 0
$Bs'b$	−1, 0	−1, 0	0, 2	0, 2
$Bs's$	−1, 0	−1, 0	0, 2	0, 2
$Nb'b$	2, 1	0, 0	2, 1	0, 0
$Nb's$	0, 0	1, 2	0, 0	1, 2
$Ns'b$	2, 1	0, 0	2, 1	0, 0
$Ns's$	0, 0	1, 2	0, 0	1, 2

Oligopolistic competition
(in strategic and extensive forms)

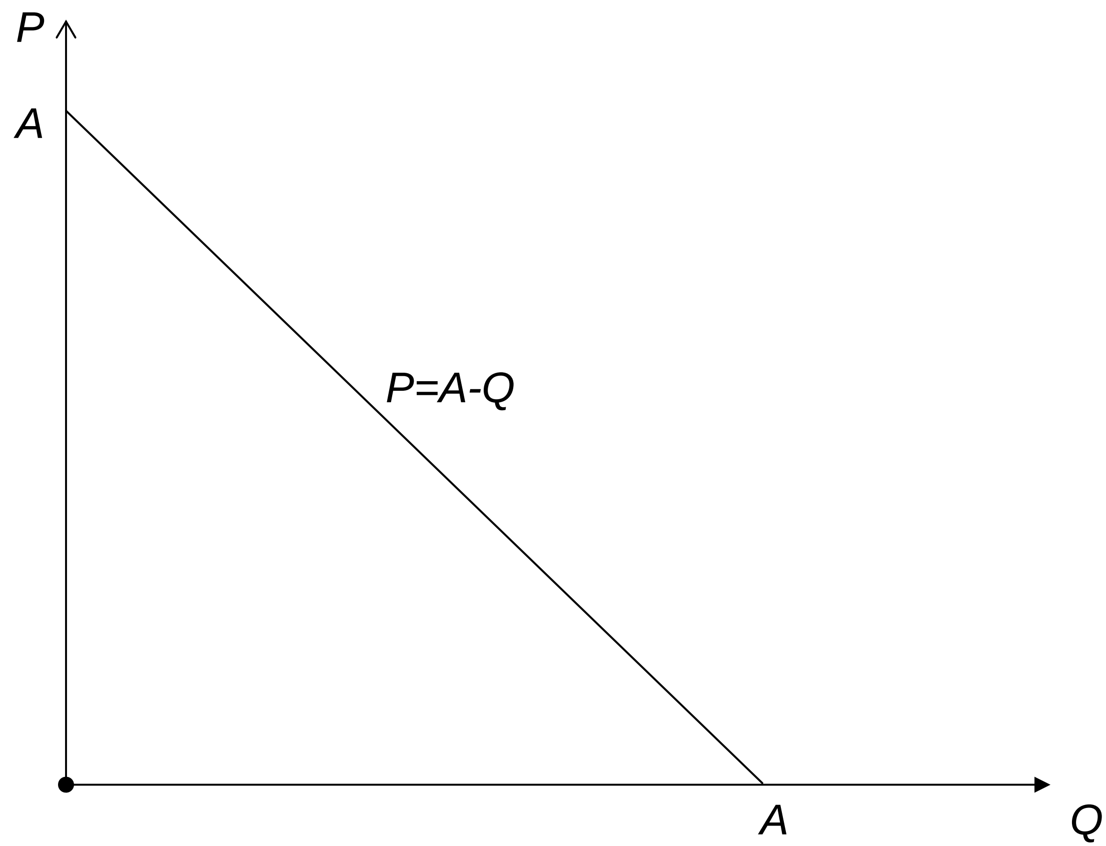
Cournot's oligopoly model (1838)

- A single good is produced by two firms (the industry is a "duopoly").
- The cost for firm $i = 1, 2$ for producing q_i units of the good is given by $c_i q_i$ ("unit cost" is constant equal to $c_i > 0$).
- If the firms' total output is $Q = q_1 + q_2$ then the market price is

  $$P = A - Q$$

  if $A \geq Q$ and zero otherwise (linear inverse demand function). We also assume that $A > c$.

The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

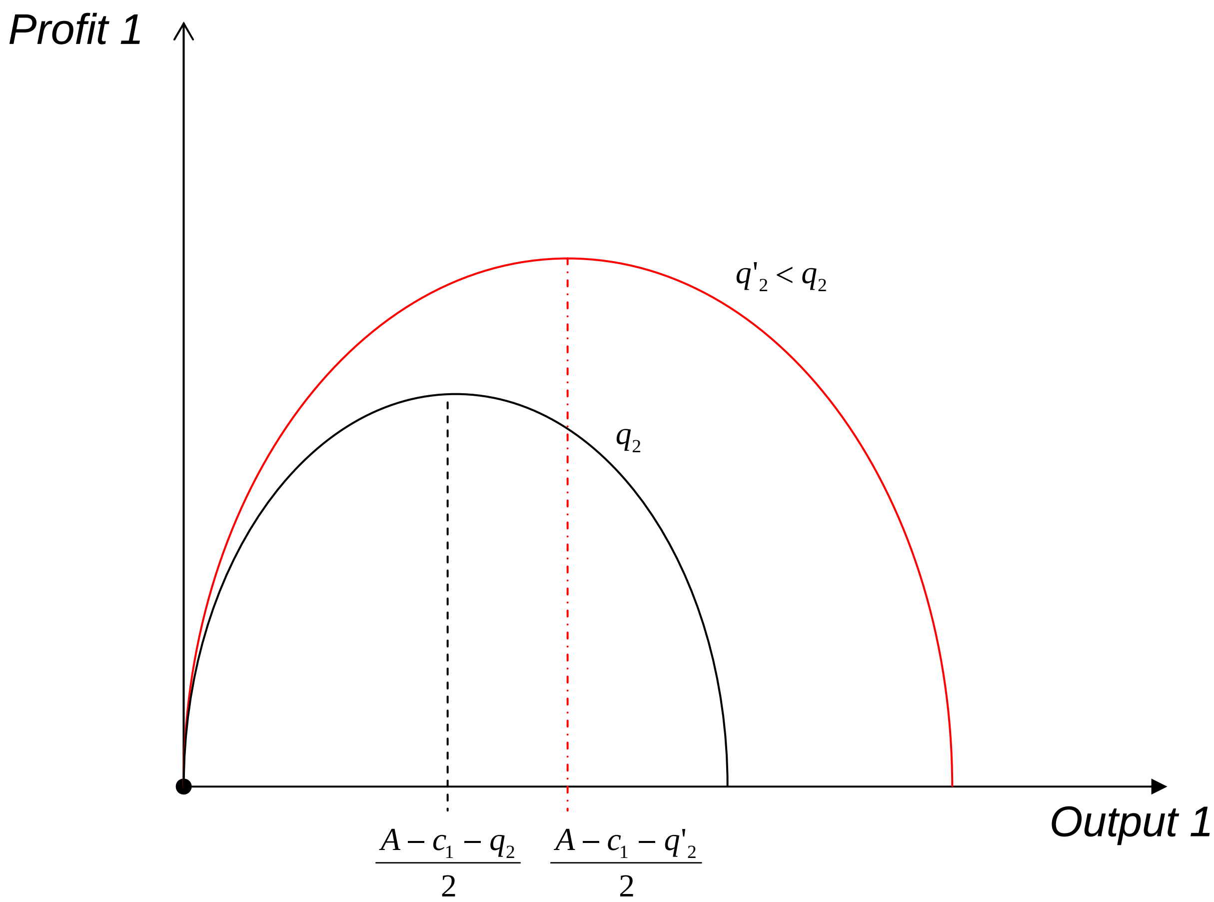
But first we need the firms payoffs (profits):

$$\begin{aligned}
\pi_1 &= Pq_1 - c_1 q_1 \\
&= (A - Q)q_1 - c_1 q_1 \\
&= (A - q_1 - q_2)q_1 - c_1 q_1 \\
&= (A - q_1 - q_2 - c_1)q_1
\end{aligned}$$

and similarly,

$$\pi_2 = (A - q_1 - q_2 - c_2)q_2$$

Firm 1's profit as a function of its output
(given firm 2's output)



To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2.

Using calculus, we set the derivative of firm 1's profit with respect to q_1 equal to zero and solve for q_1:

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the values of q_2 and c_1.

Because firm 2's cost function is $c_2 \neq c_1$, its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

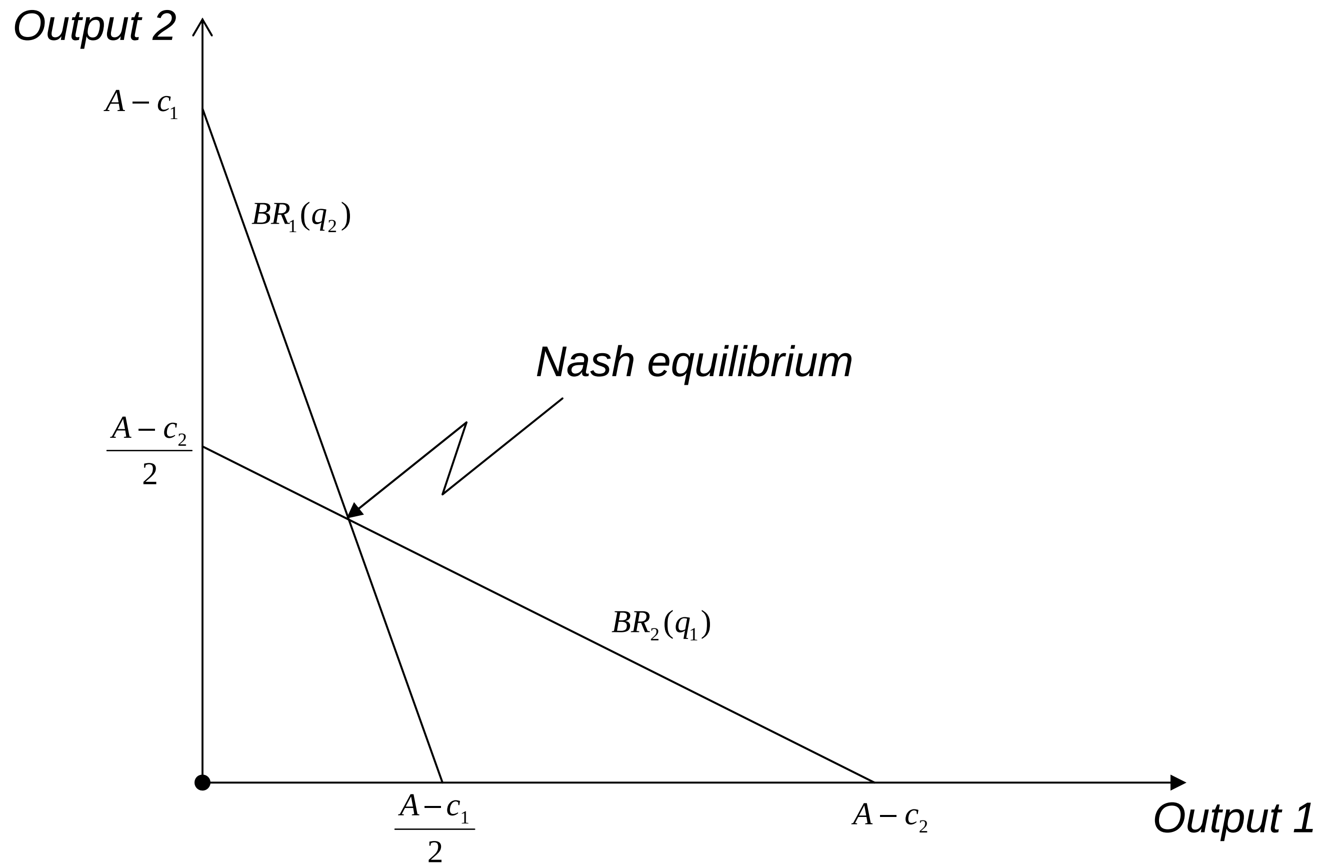
A Nash equilibrium of the Cournot's game is a pair (q_1^*, q_2^*) of outputs such that q_1^* is a best response to q_2^* and q_2^* is a best response to q_1^*.

From the figure below, we see that there is exactly one such pair of outputs

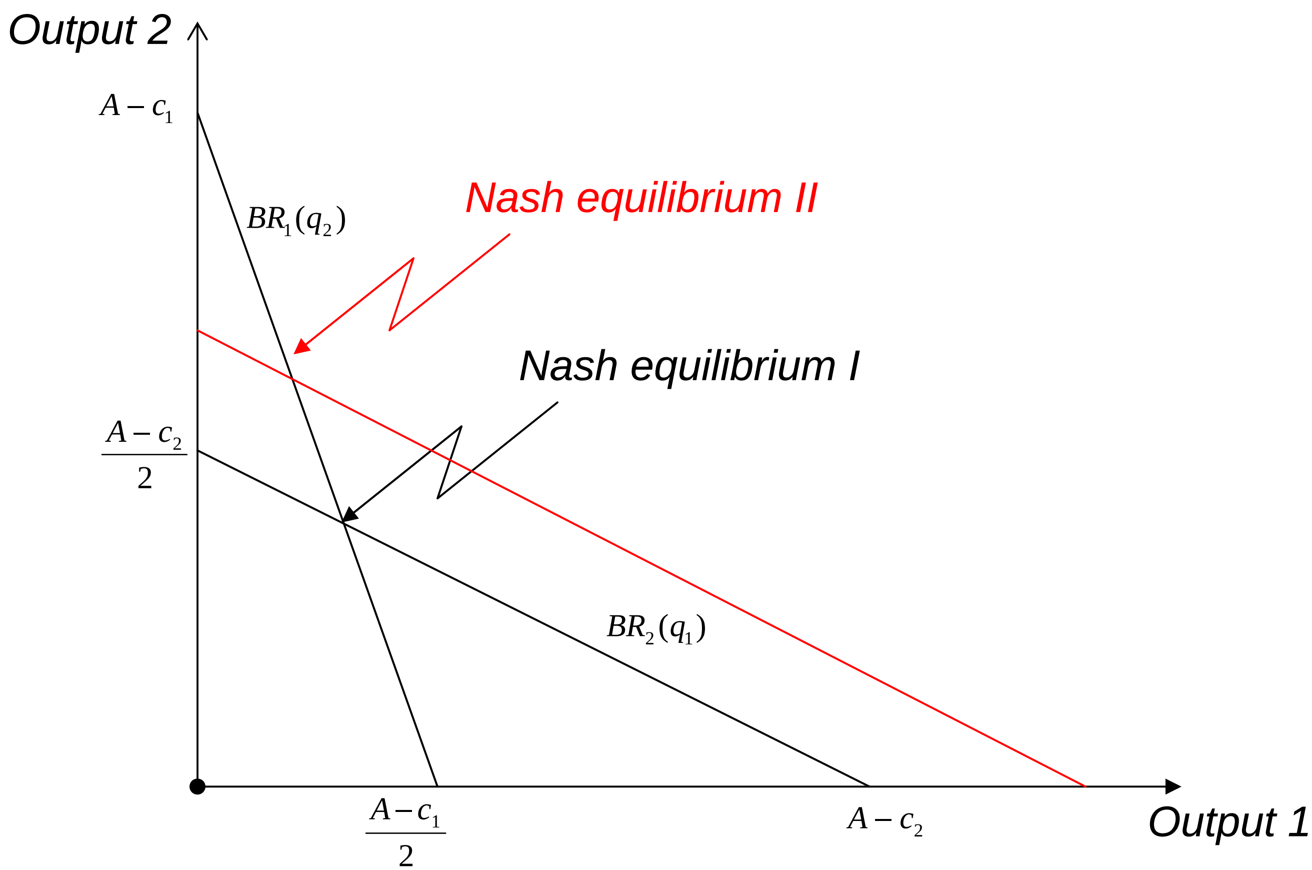
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

The best response functions in the Cournot's duopoly game



Nash equilibrium comparative statics
(a decrease in the cost of firm 2)



A question: what happens when consumers are willing to pay more (*A* increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

[1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.

[2] The relation between the equilibrium profits and the number of firms.

[1] Collusive outcomes: in the Cournot's duopoly game, there is a pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium.

[2] Competition: The price at the Nash equilibrium if the two firms have the *same* unit cost $c_1 = c_2 = c$ is given by

$$\begin{aligned} P^* &= A - q_1^* - q_2^* \\ &= \frac{1}{3}(A + 2c) \end{aligned}$$

which is above the unit cost c. But as the number of firm increases, the equilibrium price deceases, approaching c (zero profits!).

Stackelberg's duopoly model (1934)

How do the conclusions of the Cournot's duopoly game change when the firms move sequentially? Is a firm better off moving before or after the other firm?

Suppose that $c_1 = c_2 = c$ and that firm 1 moves at the start of the game. We may use backward induction to find the subgame perfect equilibrium.

– First, for *any* output q_1 of firm 1, we find the output q_2 of firm 2 that maximizes its profit. Next, we find the output q_1 of firm 1 that maximizes its profit, *given the strategy* of firm 2.

Firm 2

Since firm 2 moves after firm 1, a strategy of firm 2 is a *function* that associate an output q_2 for firm 2 for each possible output q_1 of firm 1.

We found that under the assumptions of the Cournot's duopoly game Firm 2 has a unique best response to each output q_1 of firm 1, given by

$$q_2 = \frac{1}{2}(A - q_1 - c)$$

(Recall that $c_1 = c_2 = c$).

Firm 1

Firm 1's strategy is the output q_1 the maximizes

$$\pi_1 = (A - q_1 - q_2 - c)q_1 \quad \text{subject to} \quad q_2 = \tfrac{1}{2}(A - q_1 - c)$$

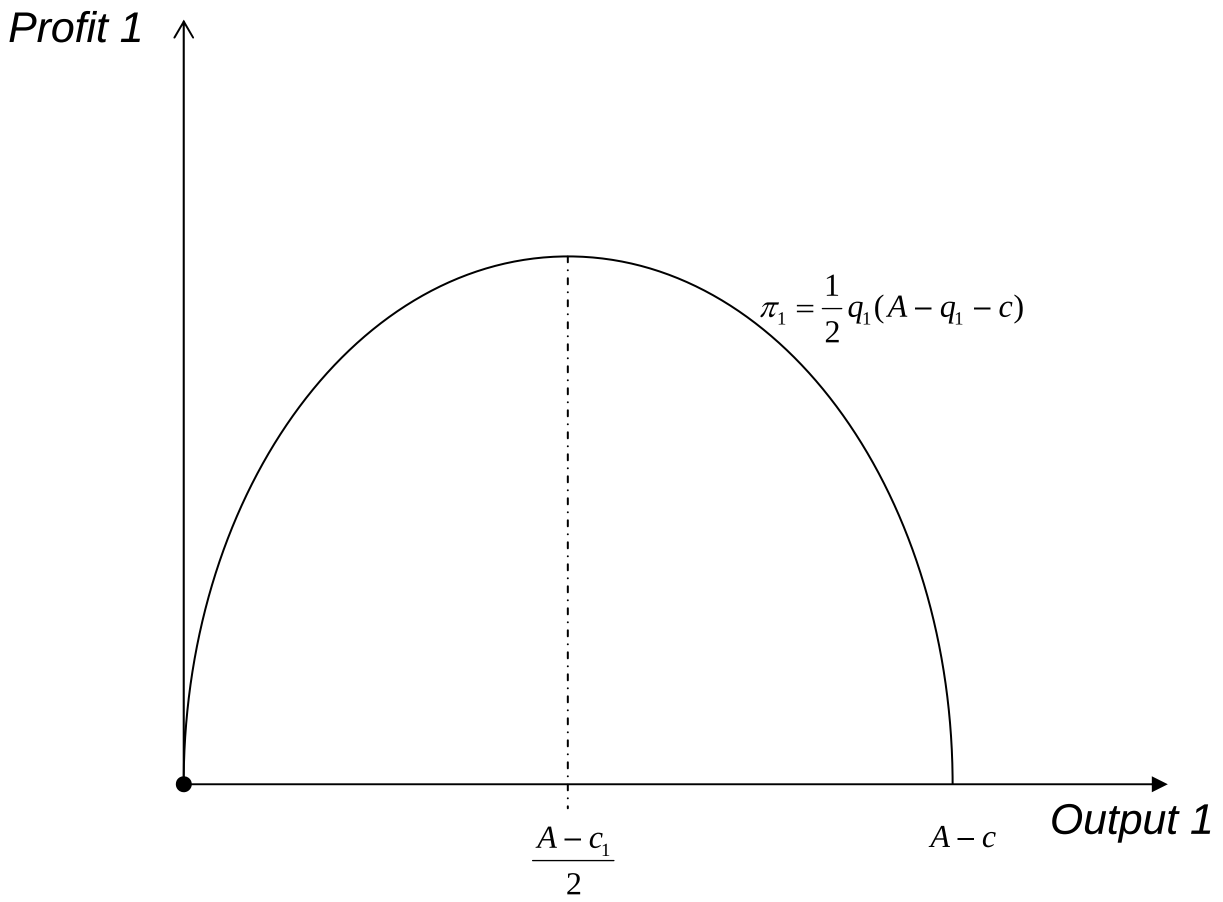
Thus, firm 1 maximizes

$$\pi_1 = (A - q_1 - (\frac{1}{2}(A - q_1 - c)) - c)q_1 = \frac{1}{2}q_1(A - q_1 - c).$$

This function is quadratic in q_1 that is zero when $q_1 = 0$ and when $q_1 = A - c$. Thus its maximizer is

$$q_1^* = \frac{1}{2}(A - c).$$

Firm 1's (first-mover) profit in Stackelberg's duopoly game



We conclude that Stackelberg's duopoly game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output

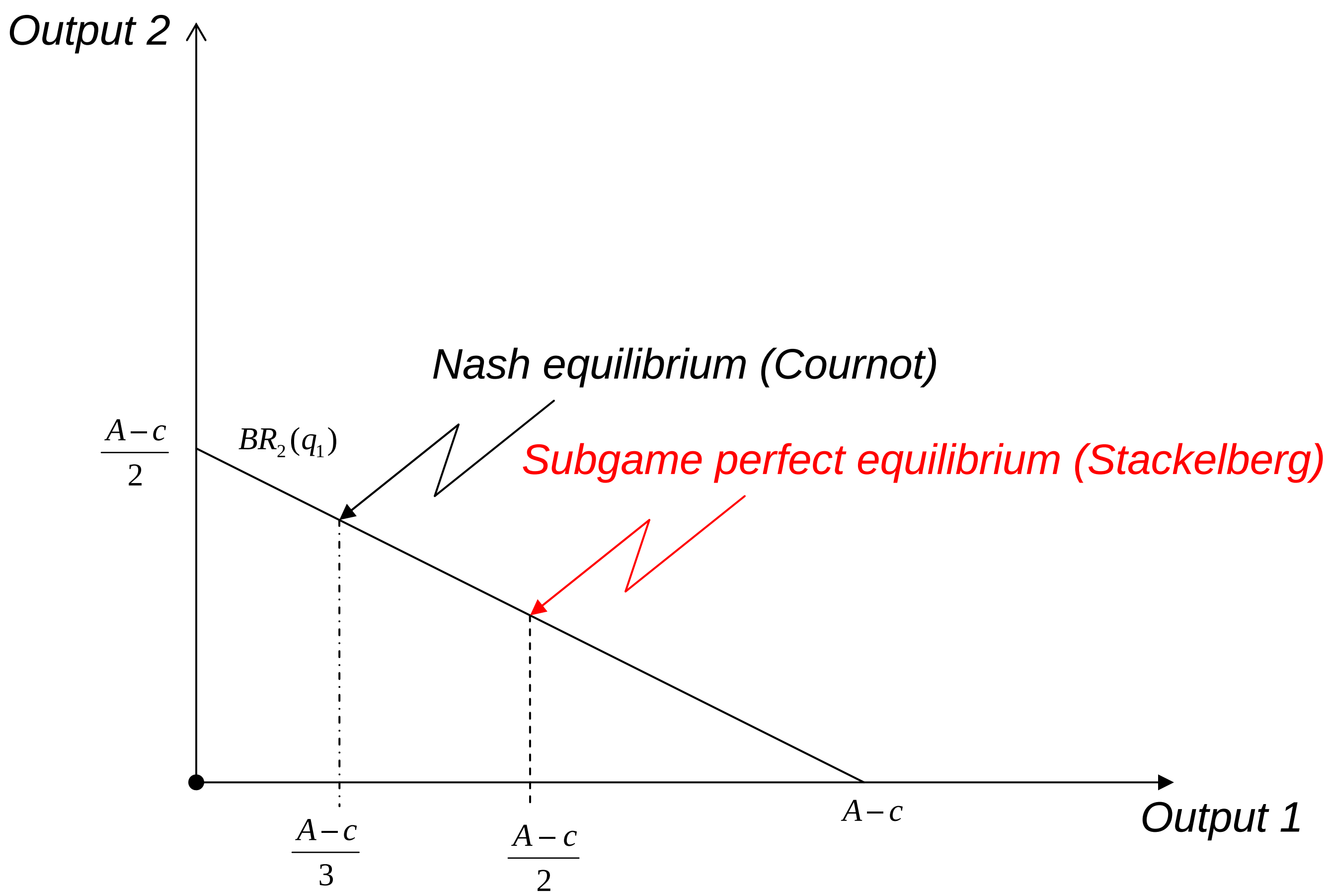
$$q_1^* = \frac{1}{2}(A - c)$$

and firm 2's output is

$$\begin{aligned} q_2^* &= \frac{1}{2}(A - q_1^* - c) \\ &= \frac{1}{2}(A - \frac{1}{2}(A - c) - c) \\ &= \frac{1}{4}(A - c). \end{aligned}$$

By contrast, in the unique Nash equilibrium of the Cournot's duopoly game under the same assumptions ($c_1 = c_2 = c$), each firm produces $\frac{1}{3}(A - c)$.

The subgame perfect equilibrium of Stackelberg's duopoly game



Auctions

Types of auctions

Sequential / simultaneous

Bids may be called out sequentially or may be submitted simultaneously in sealed envelopes:

- English (or oral) – the seller actively solicits progressively higher bids and the item is soled to the highest bidder.

- Dutch – the seller begins by offering units at a “high” price and reduces it until all units are soled.

- Sealed-bid – all bids are made simultaneously, and the item is sold to the highest bidder.

First-price / second-price

The price paid may be the highest bid or some other price:

- <u>First-price</u> – the bidder who submits the highest bid wins and pay a price equal to her bid.

- <u>Second-prices</u> – the bidder who submits the highest bid wins and pay a price equal to the second highest bid.

<u>Variants</u>: all-pay (lobbying), discriminatory, uniform, Vickrey (William Vickrey, Nobel Laureate 1996), and more.

Private-value / common-value

Bidders can be certain or uncertain about each other's valuation:

- In private-value auctions, valuations differ among bidders, and each bidder is certain of her own valuation and can be certain or uncertain of every other bidder's valuation.

- In common-value auctions, all bidders have the same valuation, but bidders do not know this value precisely and their estimates of it vary.

First-price auction (with perfect information)

To define the game precisely, denote by v_i the value that bidder i attaches to the object. If she obtains the object at price p then her payoff is $v_i - p$.

Assume that bidders' valuations are all different and all positive. Number the bidders 1 through n in such a way that

$$v_1 > v_2 > \cdots > v_n > 0.$$

Each bidder i submits a (sealed) bid b_i. If bidder i obtains the object, she receives a payoff $v_i - b_i$. Otherwise, her payoff is zero.

Tie-breaking – if two or more bidders are in a tie for the highest bid, the winner is the bidder with the highest valuation.

In summary, a first-price sealed-bid auction with perfect information is the following strategic game:

– Players: the n bidders.

– Actions: the set of possible bids b_i of each player i (nonnegative numbers).

– Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \bar{b} & \text{if} \quad b_i = \bar{b} \text{ and } v_i > v_j \text{ if } b_j = \bar{b} \\ 0 & \text{if} \quad b_i < \bar{b} \end{cases}$$

where $\bar{b}$ is the highest bid.

The set of Nash equilibria is the set of profiles $(b_1, ..., b_n)$ of bids with the following properties:

$$\begin{array}{ll} [1] & v_2 \leq b_1 \leq v_1 \\ [2] & b_j \leq b_1 \ \underline{\text{for all}}\ j \neq 1 \\ [3] & b_j = b_1 \ \underline{\text{for some}}\ j \neq 1 \end{array}$$

It is easy to verify that all these profiles are Nash equilibria. It is harder to show that there are no other equilibria. We can easily argue, however, that there is no equilibrium in which player 1 does not obtain the object.

$\Longrightarrow$ The first-price sealed-bid auction is socially efficient, but does not necessarily raise the most revenues.

Second-price auction (with perfect information)

A second-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.

- Actions: the set of possible bids b_i of each player i (nonnegative numbers).

- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \overline{b} & \text{if} \quad b_i > \overline{b} \text{ or } b_i = \overline{b} \text{ and } v_i > v_j \text{ if } b_j = \overline{b} \\ 0 & \text{if} \quad b_i < \overline{b} \end{cases}$$

  where $\overline{b}$ is the highest bid submitted by a player other than i.

First note that for any player i the bid $b_i = v_i$ is a (weakly) dominant action (a "truthful" bid), in contrast to the first-price auction.

The second-price auction has many equilibria, but the equilibrium $b_i = v_i$ for all i is distinguished by the fact that every player's action dominates all other actions.

Another equilibrium in which player $j \neq 1$ obtains the good is that in which

[1] $b_1 < v_j$ and $b_j > v_1$
[2] $b_i = 0$ <u>for all</u> $i \neq \{1, j\}$

Common-value auctions and the winner's curse

Suppose we all participate in a sealed-bid auction for a jar of coins. Once you have estimated the amount of money in the jar, what are your bidding strategies in first- and second-price auctions?

The winning bidder is likely to be the bidder with the largest positive error (the largest overestimate).

In this case, the winner has fallen prey to the so-called the winner's curse. Auctions where the winner's curse is significant are oil fields, spectrum auctions, pay per click, and more.