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Symmetric Nash Equilibrium Bid Functions for Various Auctions

Consider a symmetric two-bidder private-value, sealed-bid auction for a single item in which the highest bidder wins the item, the first player pays $c(b_1,b_2)$, and the second player pays $c(b_2,b_1)$, with the function $c(\cdot,\cdot)$ specified by the auction rules. Assume that $c(\cdot,\cdot)$ satisfies the condition $c(0,b) = 0$, so that it is consistent with individual rationality. This class of cost functions includes standard first and second price auctions, as well as first and second price all-pay auctions in which a bidder pays, respectively, his own bid or the smaller of the two bids. Suppose the bidders' values are drawn from a known common distribution $G(v)$ that has a density $g(v)$. In a symmetric Nash equilibrium, each player will use a bid function $b = B(v)$. To analyze such a Nash equilibrium, suppose bidder 2 uses the bid function, bidding $b_2 = B(v_2)$. Then the payoff to bidder 1 from a bid b_1 is $v_1\cdot\mathbf{1}(b_1>B(v_2)) - c(b_1,B(v_2))$. Suppose the bid function $b = B(v)$ is increasing and differentiable, and let $v = V(b)$ denote its inverse. Then, the event $b_1 > B(v_2)$ occurs with probability $G(V(b_1))$. The expected payoff to bidder 1 is then

$$v_1\cdot G(V(b_1)) - \int_0^\infty c(b_1,B(v_2))g(v_2)dv_2.$$

Bidder 1 chooses b_1 to maximize this expected payoff. Then b_1 satisfies the first-order condition

$$0 = v_1\cdot g(V(b_1))V'(b_1) - \frac{d}{db_1}\int_0^\infty c(b_1,B(v_2))g(v_2)dv_2.$$

In a symmetric Nash equilibrium, the b_1 that solves this condition must be $b_1 = B(v_1)$. Substituting this in and using the identities $V(B(v_1)) = v_1$ and $V'(B(v_1)) = 1/B'(v_1)$ gives

$$B'(v_1) = v_1\cdot g(v_1)/\frac{d}{db_1}\int_0^\infty c(b_1),B(v_2))g(v_2)dv_2|_{b_1=B(v_1)}.$$

If this differential equation can be solved for $B(v)$, with the boundary condition $B(0) = 0$, then this characterizes a symmetric Nash equilibrium for the specified auction mechanism. The expected revenue to the seller is

$$R = 2\int_{v_1=0}^\infty\int_{v_2=0}^{v_1} c(B(v_1),B(v_2))g(v_1)g(v_2)dv_1dv_2. + 2\int_{v_1=0}^\infty\int_{v_2=0}^{v_1} c(B(v_1),B(v_2))g(v_1)g(v_2)dv_1dv_2.$$

The revenue equivalence theorem implies that all auctions of the form given above will have the same R, since they have the same assignment function (awarding the item to the higher value bidder with probability one) and have expected cost zero at value zero.

The following table gives some cases.

	Standard 1st Price	Standard 2nd Price	1st Price All-Pay	2nd Price All-Pay
$c(b_1,b_2)$	$b_1 \cdot 1(b_1>b_2)$	$b_2 \cdot 1(b_1>b_2)$	b_1	$\min(b_1,b_2)$
$\int_0^\infty c(b_1,B(v_2))g(v_2)dv_2$	$b_1 \cdot G(V(b_1))$	$\int_0^{V(b_1)} B(v)g(v)dv$	b_1	$\int_0^{V(b_1)} B(v)g(v)dv + b_1 \cdot [1-G(V(b_1))]$
$\frac{d}{db_1}\int_0^\infty c(b_1,B(v_2))g(v_2)dv_2$ evaluated at $b_1=B(v_1)$	$G(V(b_1)) + b_1 \cdot g(V(b_1))V'(b_1)$	$B(V(b_1))g(V(b_1))V'(b_1)$	1	$1-G(V(b_1))$
FOC at $b_1 = B(v_1)$	$B'(v) = (v-B(v))g(v)/G(v)$	$B(v) = v$	$B'(v) = vg(v)$	$B'(v) = vg(v)/[1-G(v)]$
Symmetric Nash equilibrium bid function B(v)	$\int_0^v sg(s)ds/G(v)$	v	$\int_0^v sg(s)ds$	$\int_0^v [sg(s)/(1-G(s))]ds$
Seller Expected Revenue R	$2\int_{v=0}^\infty [1-G(v)]g(v)dv$	$2\int_{v=0}^\infty [1-G(v)]g(v)dv$	$2\int_{v=0}^\infty [1-G(v)]g(v)dv$	$2\int_{v=0}^\infty [1-G(v)]g(v)dv$
B(v) when $G(v) = v$, $0<v<1$	$v/2$	v	$v^2/2$	$-\log(1-v) - v$
B(v) when $G(v) = 1-e^{-v/a}$	$(a-(v+a)e^{-v/a})/(1-e^{-v/a})$	v	$a-(v+a)e^{-v/a}$	$v^2/2a$

Verify as an exercise that the bid from a standard 2nd price auction is greater than the bid from a standard 1st price auction, the bid from a 2nd price all-pay auction is greater than the bid from a 1st price all-pay auction, and the bid from a standard 1st price auction is greater than the bid from a 1st price all-pay auction. Show that the bid from a 2nd price all-pay auction may be larger or smaller than the bid from a 1st price standard auction, depending on G and v.