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Problem Set 10. Example Final Exam Questions
(With Solutions)

1. There are J firms in an industry. Each can try to convince Congress to give the industry a subsidy. Let H_j denote the hours of effort put in by firm j, and let $w_jH_j^2$, where w_j is a positive constant, be the cost of this effort to firm j. When the effort levels of the firms are $(H_1,...,H_J)$, the value to each firm of the subsidy that gets approved is $\alpha\sum_{j=1}^{J} H_j + \beta\prod_{j=1}^{J} H_j$, where α and β are constants. Show that each firm has a strictly dominant strategy if and only if β = 0. What is this dominant strategy when β = 0?

The payoff to firm 1 is $\alpha\sum_{j=1}^{J} H_j + \beta\prod_{j=1}^{J} H_j - w_1H_1^2$, which is maximized, given the effort levels of others, at $H_1 = (\alpha + \beta\prod_{j=2}^{J} H_j)/2w_1$. This is independent of the actions of others if β = 0.

2. [Hint: difficult, use a separating hyperplane argument] Prove that in a two-player game with finite sets of actions, if action a_1 for player 1 is never a best response for any mixed strategy of player 2, then a_1 is strictly dominated by some mixed strategy for player 1.

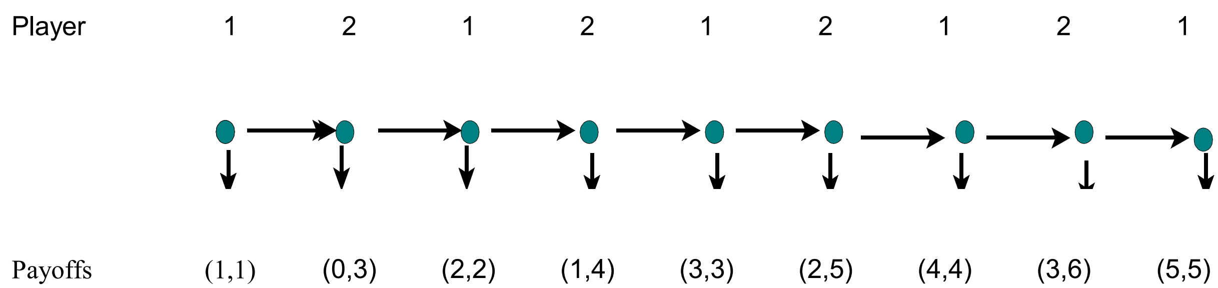
Suppose player 2 has J actions and player 1 has K actions. A mixed strategy for 2 is defined by a non-negative probability vector $p \in \mathbb{R}^J$ whose components sum to one. Associated with action k of player 1 is a vector $\pi_k \in \mathbb{R}^J$ whose component j gives the payoff to 1 when 2 plays his action j. Then, $p\cdot\pi_k$ is the expected payoff to player 1 from action k when 2 plays the mixed strategy p. Let Q denote the closed convex set containing all points q that satisfy $q \le r_1\pi_1...r_K\pi_K$, where $(r_1,...,r_K)$ is a non-negative vector whose components sum to one. Note that each π_k is either a vertex on the boundary of Q, or is contained in the interior of Q. Each point in this set can be associated with a mixed strategy for player 1 combined with possible disposal of some payoffs. If π_1 is on the northeast boundary of Q, then there exists a separating hyperplane (with normal p) that separates π_1 from the interior of Q. If player 2 chooses a mixed strategy with probability p corresponding to this normal, then player 1 maximizes $p\cdot q$ over Q at $q = \pi_1$. Conversely, if π_1 does not maximize $p\cdot q$ over Q for any mixed strategy p, then it must be interior to Q. This in turn implies that there exists a convex combination of the vertices of Q that is strictly to the northeast of π_1. This in turn can be interpreted as a mixed strategy that strictly dominates π_1.

3. Consider a bargaining game in which two firms are considering a joint venture that will earn a profit of one million dollars, but they must agree on how to split the profit. Bargaining works as follows: Each makes a bid for a share of the one million. If the sum of the two bids exceeds one million, then the bargaining fails and both get nothing. If the sum of the two bids is less than one million, each gets his bid and the remainder goes to charity What are each player's strictly dominated strategies? What are their weakly dominated strategies? What are the pure strategy Nash equilibria for this game?

Every pair of bids (b_1,b_2) that sum to one million and are positive for each firm define a Nash equilibrium with the property that given the bid of the rival firm, the difference between one million and that bid is a unique optimal response. In particular, a bid of ½ million from each firm is the unique symmetric Nash equilibrium. Only the strategy of bidding zero is weakly dominated.

4. The centipede game for two players works as follows: Each player starts with one dollar. Starting with player 1, they alternately say “Stop” or “Continue”. When a player says “Continue”, one dollar is taken from her pile, and two dollars are added to her opponent’s pile. As soon as either player says “Stop”, the game terminates, and each receives the dollars in her own pile. Alternately, the game stops if each player’s pile reaches five dollars. (A) Draw the extensive form of this game. (B) Find the unique subgame perfect Nash equilibrium.

Apply backward induction to centipede game to show that “Stop” is the dominant strategy at every node. Its extensive form is given below:



5. Two firms, 1 and 2, have competitive products in the same market, with prices p_1 and p_2, and marginal costs $m_1 < m_2$. The demands for goods 1and 2 are

$$x_1 = \min\{2, 1 - \beta(p_1 - p_2)\},$$
$$x_2 = \max\{0, 1 - \beta(p_2 - p_1)\}.$$

Find Nash equilibrium price strategies for the two firms. Be sure to include the possibilities that both firms produce, or only firm 1 produces, with firm 2 poised to enter under profitable conditions.

The payoff to firm 1 is $\pi_1 = (p_1 - m_1)x_1$, and to firm 2 is $\pi_2 = (p_2 - m_2)x_2$. In Nash equilibrium, each chooses its price to maximize payoff, given the price of the other. Each maximization gives a reaction function that maps rival’s price into your own price. The Nash equilibrium occurs at the intersection of these reaction functions. The only tricky part is handling the boundary constraints. The simplest approach is to first solve ignoring the boundaries, obtaining $p_1 = (2m_1 + m_2)/3 + 1/\beta$, $p_2 = (m_1 + 2m_2)/3 + 1/\beta$, $x_1 = 1 - \beta(m_1 - m_2)/3$, $x_2 = 1 - \beta(m_2 - m_1)/3$. If $\beta(m_2 - m_1)/3 \le 1$, this solution is consistent with the constraints. Otherwise, the corner solution is $p_2 = m_2$, $p_1 = m_2 - 1/\beta$, $x_1 = 2$, $x_2 = 0$.

6. Two identical objects are to be auctioned in sequence to three symmetric bidders whose independent private values for an object are drawn from a known common distribution G(v), using English auctions. (The value of a second object to a single bidder is zero.) Show that a subgame perfect Nash equilibrium assigns the two objects to the bidders with the two highest values, and the price paid by each will be the third highest value.

Suppose players 2 and 3 each follow the strategy: “Stay in the initial auction as long as no one has dropped and the price is below my value, drop first when the price reaches my value, and if a rival is the first to drop, then drop second at a price ε above the price at the first drop.” Consider the optimal response of player 1 to these strategies. Player 1 will choose a bid $b_1 \le v_1$ at which he would drop first in the initial auction. If $b_1 < v_1$, then the possible cases are (1) $\min(v_2, v_3) < b_1$, (2) $b_1 \le \min(v_2, v_3) \le v_1$, and (3) $v_1 < \min(v_2, v_3)$. In case (1), one rival drops first, the second drops immediately, and player 1 wins the initial auction with payoff v_1 - min(v_2,v_3). Further, player 1 would never bid higher than min(v_2,v_3)+ε, since he knows he can win the final auction with payoff v_1 - min(v_2,v_3). In case (3), player 1 drops first, and also loses the final auction. In case (2), player 1 drops first, a rival wins the first auction at price $b_1 \le$ min(v_2,v_3), and then in the final auction player 1 faces with probability ½ the rival with value min(v_2,v_3) $\le v_1$ which player 1 will win with payoff v_1 - min(v_2,v_3), and with probability ½ the rival with value max(v_2,v_3), resulting in payoff to player 1 of max(v_1 - max(v_2,v_3),0). Comparing (1) and (2), a strategy with $b_1 < v_1$ is dominated by the strategy with $b_1 = v_1$. Then, the strategy described above is a Nash equilibrium, and in this equilibrium the objects go to the two highest-value players at the third-highest value.

7. Each respondent in a poll of N+1 consumers is asked to submit a bid giving the maximum cost to them at which they would vote "Yes" for provision of a particular public good, and told that the good will be provided if a majority of those polled are in favor at the actual cost per capita C that provision of the good turns out to require. It is emphasized that the actual cost per capita will be charged to everyone if and only if the good is provided, so that in particular there is no linkage from their bid to the amount they will have to pay if the good is provided. Is this mechanism incentive-compatible, so that each consumer bids her true value?

Consider respondent 1. This respondent has a positive probability p(C) of being <u>decisive</u> when the actual cost is C, that is being the voter who tips the result for or against. Let g(C) be the probability density of actual cost per capita, v be the respondent's value, s be the respondent's bid. Then, the expected payoff to this respondent is

$$\int_0^\infty (v-C)1(C<s)p(C)g(C)dC = \int_0^s (v-C)p(C)g(C)dC.$$

This is maximized in s at s = v, so that the mechanism is truth-revealing, provided p(s) > 0 and g(s) > 0.