Problem Set 2
Suggested Solutions

1.
Following the notation in Appendix A.2 of D. McFadden "Definite Quadratic Forms Subject to Constraints" in M. Fuss and D. McFadden Production Economics, Vol. 1, we have for the $(n+m)$x$(n+m)$ Hessian matrix of the Lagrangean[1]:

$$D^2L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdot & \frac{\partial^2 L}{\partial x_n x_1} & \frac{\partial^2 L}{\partial p_1 x_1} & \cdot & \frac{\partial^2 L}{\partial p_m x_1} \\ \vdots & \cdot & \vdots & \vdots & \cdot & \vdots \\ \frac{\partial^2 L}{\partial x_1 x_n} & \cdot & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial^2 L}{\partial p_1 x_n} & \cdot & \frac{\partial^2 L}{\partial p_m x_n} \\ \frac{\partial^2 L}{\partial x_1 p_1} & \cdot & \frac{\partial^2 L}{\partial x_n p_1} & \frac{\partial^2 L}{\partial p_1^2} & \cdot & \frac{\partial^2 L}{\partial p_m p_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 L}{\partial x_1 p_m} & \cdot & \frac{\partial^2 L}{\partial x_n p_m} & \frac{\partial^2 L}{\partial p_1 p_m} & \cdot & \frac{\partial^2 L}{\partial p_m^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdot & \frac{\partial^2 L}{\partial x_n x_1} & \frac{\partial g^1}{\partial x_1} & \cdot & \frac{\partial g^m}{\partial x_1} \\ \vdots & \cdot & \vdots & \vdots & \cdot & \vdots \\ \frac{\partial^2 L}{\partial x_1 x_n} & \cdot & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial g^1}{\partial x_n} & \cdot & \frac{\partial g^m}{\partial x_n} \\ \frac{\partial g^1}{\partial x_1} & \cdot & \frac{\partial g^1}{\partial x_n} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g^m}{\partial x_1} & \cdot & \frac{\partial g^m}{\partial x_n} & 0 & \cdot & 0 \end{bmatrix}$$

[1] As you will notice from the proof that follows, there was a typo in the question. $L_{x,p}$ was meant to be the Hessian of the Lagrangean for which a more appropriate notation is D^2L. My true apologies for any inconvenience and frustration this might have caused you in attempting this question.

Consider that the $n \times m$ matrix $G_x = \begin{bmatrix} \frac{\partial g^1}{\partial x_1} & . & . & \frac{\partial g^m}{\partial x_1} \\ . & & & \\ . & & & \\ \frac{\partial g^1}{\partial x_n} & . & . & \frac{\partial g^m}{\partial x_n} \end{bmatrix}$ does not have maximal rank:

rank $G_x < m$.

Then its columns are linearly dependent. Hence, there will be at least one column, say column *j*, that can be expressed as a linear combination of the other $m-1$ columns.

$$\begin{bmatrix} \frac{\partial g^j}{\partial x_1} \\ . \\ . \\ \frac{\partial g^j}{\partial x_n} \end{bmatrix} = \sum_i \lambda_i \begin{bmatrix} \frac{\partial g^i}{\partial x_1} \\ . \\ . \\ \frac{\partial g^i}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \sum_i \lambda_i \frac{\partial g^i}{\partial x_1} \\ . \\ . \\ \sum_i \lambda_i \frac{\partial g^i}{\partial x_1} \end{bmatrix} \quad \text{for } i \in \{1,2,...,m\} - \{j\}$$

At the critical point $(\bar{x}, \bar{p})$, we have:

$$G_x = \begin{bmatrix} \left.\frac{\partial g^1}{\partial x_1}\right|_{(\bar{x},\bar{p})} & . & . & \left.\frac{\partial g^m}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ . & & & . \\ . & & & . \\ \left.\frac{\partial g^1}{\partial x_n}\right|_{(\bar{x},\bar{p})} & . & . & \left.\frac{\partial g^m}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix} = \begin{bmatrix} \left.\frac{\partial g^1}{\partial x_1}\right|_{(\bar{x},\bar{p})} & \sum_{-j} \lambda_i \left.\frac{\partial g^i}{\partial x_1}\right|_{(\bar{x},\bar{p})} & \left.\frac{\partial g^m}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ & & \\ \left.\frac{\partial g^1}{\partial x_n}\right|_{(\bar{x},\bar{p})} & \sum_{-j} \lambda_i \left.\frac{\partial g^i}{\partial x_n}\right|_{(\bar{x},\bar{p})} & \left.\frac{\partial g^m}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix}$$

Using the above, substitute for the (*n*+*j*)th row and column of the Hessian matrix.

$$D^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & . & \frac{\partial^2 L}{\partial x_n x_1} & \frac{\partial g^1}{\partial x_1} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & \frac{\partial g^m}{\partial x_1} \\ . & . & . & . & . & . \\ \frac{\partial^2 L}{\partial x_1 x_n} & . & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial g^1}{\partial x_n} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & \frac{\partial g^m}{\partial x_n} \\ \frac{\partial g^1}{\partial x_1} & . & \frac{\partial g^1}{\partial x_n} & 0 & . & 0 \\ \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & . & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & . & . & . \\ \frac{\partial g^m}{\partial x_1} & . & \frac{\partial g^m}{\partial x_n} & 0 & . & 0 \end{bmatrix}$$

But, by its definition, at the critical point $(\bar{x}, \bar{p})$, we have:

$$L_x(\bar{x}, \bar{p}) = 0 \Leftrightarrow$$
$$[F_x + G_x p] = 0 \Leftrightarrow$$
$$\begin{bmatrix} \left.\dfrac{\partial F}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ \cdot \\ \cdot \\ \left.\dfrac{\partial F}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix} + \begin{bmatrix} \left.\dfrac{\partial g^1}{\partial x_1}\right|_{(\bar{x},\bar{p})} & \cdot & \cdot & \left.\dfrac{\partial g^m}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \left.\dfrac{\partial g^1}{\partial x_n}\right|_{(\bar{x},\bar{p})} & \cdot & \cdot & \left.\dfrac{\partial g^m}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix} p = 0 \Leftrightarrow$$
$$\begin{bmatrix} \left.\dfrac{\partial F}{\partial x_1} + \sum_{j=1}^{m} p_j \dfrac{\partial g^j}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ \cdot \\ \cdot \\ \left.\dfrac{\partial F}{\partial x_n} + \sum_{j=1}^{m} p_j \dfrac{\partial g^j}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix} = 0 \Leftrightarrow$$
$$\begin{bmatrix} \left.\dfrac{\partial F}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ \cdot \\ \cdot \\ \left.\dfrac{\partial F}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix} = -\begin{bmatrix} \left.\sum_{j=1}^{m} p_j \dfrac{\partial g^j}{\partial x_1}\right|_{(\bar{x},\bar{p})} \\ \cdot \\ \cdot \\ \left.\sum_{j=1}^{m} p_j \dfrac{\partial g^j}{\partial x_n}\right|_{(\bar{x},\bar{p})} \end{bmatrix}$$

Consequently, evaluated at the point $(\bar{x}, \bar{p})$, the Hessian matrix of the Lagrangean can be written as:

$$D^2 L =$$

$$\begin{bmatrix} -\sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_1^2} + \sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_1^2} & . & -\sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_n x_1} + \sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_n x_1} & \frac{\partial g^1}{\partial x_1} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & \frac{\partial g^m}{\partial x_1} \\ . & . & . & . & . & . \\ -\sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_1 x_n} + \sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_1 x_n} & . & -\sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_n^2} + \sum_{j=1}^{m} p_j \frac{\partial g^j}{\partial x_n^2} & \frac{\partial g^1}{\partial x_n} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & \frac{\partial g^m}{\partial x_n} \\ \frac{\partial g^1}{\partial x_1} & . & \frac{\partial g^1}{\partial x_n} & 0 & . & 0 \\ \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & . & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & . & . & . \\ \frac{\partial g^m}{\partial x_1} & . & \frac{\partial g^m}{\partial x_n} & 0 & . & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & . & 0 & \frac{\partial g^1}{\partial x_1} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & \frac{\partial g^m}{\partial x_1} \\ . & . & . & . & . & . \\ 0 & . & 0 & \frac{\partial g^1}{\partial x_n} & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & \frac{\partial g^m}{\partial x_n} \\ \frac{\partial g^1}{\partial x_1} & . & \frac{\partial g^1}{\partial x_n} & 0 & . & 0 \\ \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_1} & . & \sum_{-j} \lambda_i \frac{\partial g^i}{\partial x_n} & . & . & . \\ \frac{\partial g^m}{\partial x_1} & . & \frac{\partial g^m}{\partial x_n} & 0 & . & 0 \end{bmatrix}$$

This matrix has zero determinant because its (n+j)th row (column) is a linear combination of the rest (n+m-1) rows (columns).

<u>Note:</u>
The importance of this result is explained in Question 5/Part (b) below (pp. 12). In the Lagrange theorem we require that the Jacobian matrix of the constraints $DG = G_x$ has full rank (i.e. full column rank since we always consider the constraints to be less that the choice variables $k \leq n$ in these problems). This requirement is called the Non-Degeneracy Condition (NDGC) and is needed in the proof of the theorem to ensure the matrix $\underset{(nxm)}{G_x}$ has an invertible (mxm) sub-matrix so that the vector p of Lagrangean multipliers in the first order condition that defines the LCP:

$$F_x + G_x p = 0$$

indeed exists.

2. Consider the Hessian of the given function: $D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2a & 2c \\ 2c & 2b \end{bmatrix}$

For *f* to be concave, a necessary and sufficient condition is for the Hessian to be negative definite.
The leading principal minors of $D^2 f$ are given:

$M_1 = 2a$

$M_2 = \begin{vmatrix} 2a & 2c \\ 2c & 2b \end{vmatrix} = 2\left(ab - c^2\right)$

We require

$a < 0,\ ab - c^2 > 0,\ d \in R \Leftrightarrow a < 0,\ b < 0,\ ab > c^2,\ d \in R$ (I)

Note:
Recall from Problem Set 1/Question 4 that a necessary condition for a square, symmetric real matrix to be negative definite is for its diagonal elements to be all negative. This is clearly satisfied by (I).

3. Consider some $\lambda \in [0,1]$ and any $x, y \in D \subseteq R^n$.

By *D* being convex, $\lambda x + (1-\lambda) y \in D$.

The following shows that $f : D \to R$ is indeed convex.

$$\begin{aligned} & f\left(\lambda x + (1-\lambda) y\right) \\ &= \sup_i f_i\left(\lambda x + (1-\lambda) y\right) \\ &= f_{\bar{i}}\left(\lambda x + (1-\lambda) y\right) \\ &\le \lambda f_{\bar{i}}(x) + (1-\lambda) f_{\bar{i}}(y) \\ &\le \lambda \sup_i f_i(x) + (1-\lambda) \sup_i f_i(y) \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

Note that the third line uses the fact that each function $f_i : D \to R$ is bounded on *D*. Therefore, since $\forall x \in D$, $f_i(x)$ is some real number for each function $f_i(.)$, $\sup_i f_i\left(\lambda x + (1-\lambda) y\right)$ exists. Let it correspond to some $f_{\bar{i}} : D \to R$ for some $\bar{i} \in I$.

The fourth line uses the fact that all functions $f_i : D \to R,\ i \in I$ (and, hence $f_{\bar{i}} : D \to R$ also) are convex.

With respect to the function g, an attempt towards a similar proof breaks down on the fifth line as the following shows. This function cannot be convex unless $\inf_i f_i(x)$ gives the same function $f_{i^*}: D \to R$ over all $x \in D$ - in which case, the sign of the fifth line below becomes an equality and the convexity proof holds.

$$\begin{aligned}
&g(\lambda x+(1-\lambda)y)\\
&=\inf_i f_i(\lambda x+(1-\lambda)y)\\
&=f_{i^*}(\lambda x+(1-\lambda)y)\\
&\le \lambda f_{i^*}(x)+(1-\lambda)f_{i^*}(y)\\
&\ge \lambda \inf_i f_i(x)+(1-\lambda)\inf_i f_i(y)\\
&=\lambda g(x)+(1-\lambda)g(y)
\end{aligned}$$

Note again that the third line uses the fact that any function $f_i: D \to R$ is bounded on D. Therefore, $\inf_i f_i(\lambda x+(1-\lambda)y)$ exists and let it correspond to some $f_{i^*}: D \to R$ for some $i^* \in I$.

The fourth line uses the fact that all functions $f_i: D \to R,\ i \in I$ (and, hence $f_{i^*}: D \to R$ also) are convex.

4.

a. Let $u(c)=\begin{pmatrix} u(c_1)\\ .\\ u(c_T)\end{pmatrix}$ $c=\begin{pmatrix} c_1\\ .\\ c_T\end{pmatrix}$, $z=\begin{pmatrix} x_1\\ .\\ x_T\end{pmatrix}$ and $b=\begin{pmatrix} x\\ f(x_1)\\ .\\ f(x_{T-1})\end{pmatrix}$

The given optimization problem:

$$\max_{c_1,c_2..,c_T} \sum_{t=1}^{T} u(c_t) = \max_{c_1,c_2..,c_T} \begin{pmatrix} u(c_1)\\ .\\ .\\ u(c_T)\end{pmatrix}^T \begin{pmatrix} 1\\ .\\ .\\ 1\end{pmatrix}$$

$s.t.$

$c+z=b$

$c \ge 0$

$z \ge 0$

b. In this problem we have T c-type and T x-type choice variables, T equality constraints and $2T$ non-negativity constraints. Let:

$g^1(c_1, x_1) = c_1 + x_1 - x$ and $g^j(c_j, x_{j-1}, x_j) = c_j + x_j - f(x_{j-1})$ for $j = 2,...,T$.

We need to employ $3T$ Lagrangean multipliers: $p = \begin{pmatrix} \lambda_1 \\ . \\ \lambda_T \\ \mu_1 \\ . \\ \mu_T \\ \rho_1 \\ . \\ \rho_T \end{pmatrix} \in R^{3T}$

The Lagrangean function:

$$L(c_1 .., c_T, x_1,.., x_T, \lambda_1,..\lambda_T, \mu_1,.., \mu_T, \rho_1,.., \rho_T) =$$
$$\sum_{t=1}^{T} u(c_t) + \sum_{t=1}^{T} \lambda_t g^t(c_t, x_{t-1}, x_t) + \sum_{t=1}^{T} \mu_t c_t + \sum_{t=1}^{T} \rho_t x_t$$

The first-order conditions:

<u>Set I</u>

$$\frac{\partial L}{\partial c_t} = 0 \Leftrightarrow$$
$$u'(c_t) + \lambda_t \frac{\partial}{\partial c_t} g^t(c_t, x_{t-1}, x_t) + \mu_t = 0 \Leftrightarrow \qquad \text{for } t = 1,...,T \qquad (1)$$
$$u'(c_t) + \lambda_t + \mu_t = 0$$

$$\frac{\partial L}{\partial x_t} = 0 \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \lambda_t \dfrac{\partial}{\partial x_t} g^t(c_t, x_{t-1}, x_t) + \lambda_{t+1} \dfrac{\partial}{\partial x_t} g^{t+1}(c_t, x_t, x_{t+1}) + \rho_t = 0 \\ \lambda_1 c_1 + \rho_1 = 0 \end{cases} \qquad \text{for } t = 2,..T \qquad (2)$$
$$\Leftrightarrow \begin{cases} \lambda_t (c_t - f(x_{t-1})) + \lambda_{t+1} f'(x_t) + \rho_t = 0 \\ \lambda_1 c_1 + \rho_1 = 0 \end{cases}$$

Set II

$$\frac{\partial L}{\partial \lambda_t} = 0 \Leftrightarrow g^t\left(c_t, x_{t-1}, x_t\right) = 0 \qquad \text{for } t = 1,...,T \qquad (3)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial \mu_t} \geq 0 \\ \mu_t \geq 0 \\ \mu_t \dfrac{\partial L}{\partial \mu_t} = 0 \end{array}\right. \quad \Leftrightarrow \quad \begin{array}{l} c_t \geq 0 \\ \mu_t \geq 0 \\ \mu_t c_t = 0 \end{array} \qquad \text{for } t = 1,...,T \qquad (4)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial \rho_t} \geq 0 \\ \rho_t \geq 0 \\ \rho_t \dfrac{\partial L}{\partial \rho_t} = 0 \end{array}\right. \quad \Leftrightarrow \quad \begin{array}{l} x_t \geq 0 \\ \rho_t \geq 0 \\ \rho_t x_t = 0 \end{array} \qquad \text{for } t = 1,...,T \qquad (5)$$

Note that the sets of equations (1) and (2) above define a $2Tx2T$ system of equations.

c. For the sufficiency conditions see the theorem of the second-order conditions described in part (c) of the next question.

Note, however, that, in this setting, in the case where the function *f* is linear then, because all of the equality constraints are linear in the choice variables $c_1,..,c_T, x_1,..,x_T$, it suffices for the objective function $u(c) = \sum_{t=1}^{T} u(c_t)$ to be quasi-concave. This would be ensured, for example, if the elementary function $u(.)$ is quasi-concave in its argument.

5.

a. The consumer's optimization problem:

$$\max_{q_1, q_2} u(q_1, q_2) = \ln q_1 + \ln q_2$$
$$s.t.$$
$$p_1(q_1)q_1 + p_2(q_2)q_2 \leq I$$
$$q_i \geq 0 \quad i = 1,2$$

b. *The Weierstrass Theorem*:
Let $D \subseteq R^n$ be a compact set and $u : D \to R$ a continuous function on *D*.
Then *u* attains both a maximum and a minimum on *D*.
That is, $\exists\ \underline{q}, \overline{q} \in D : u(\underline{q}) \leq u(q) \leq u(\overline{q})$ for any $q \in D$

The consumption set $D = R_+^2 \cap \{(q_1, q_2) \in R^2 : p_1(q_1)q_1 + p_2(q_2)q_2 \leq I\}$ is bounded below by the point $(0,0)$ and above by the line defined by the two points $(\tilde{q}_1, 0) \in R^2 : p_1(\tilde{q}_1) = \frac{I}{\tilde{q}_1}$ and $(0, \tilde{q}_2) \in R^2 : p_2(\tilde{q}_2) = \frac{I}{\tilde{q}_2}$. Hence, it is bounded.
Note that since p_i is strictly increasing, it is injective.
Hence, $q_1 > \tilde{q}_1 \Rightarrow p_1(q_1) > p_1(\tilde{q}_1) \Rightarrow \frac{I}{p_1(q_1)} < \frac{I}{p_1(\tilde{q}_1)}$ and similarly if one starts with some $q_i < \tilde{q}_i$. Which shows that for each pair $p_i : \ i = 1,2$ the two points $(\tilde{q}_1, 0), (0, \tilde{q}_2)$ given above are unique.

The set *D* is clearly closed (see Fig. 5.I – for a given pair of functions $p_i : \ i = 1,2$ the budget set *D* is the triangular area shown which is bounded and, by including its border lines, also closed).

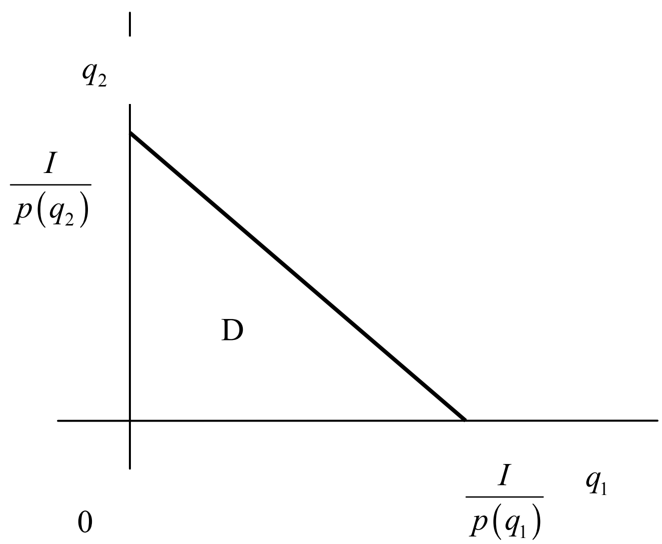

Fig. 5.I

The Lagrange Theorem:

Let $u : R^n \to R$ and $g^i : R^n \to R$ with $i = 1,2,..,k$ be C^1 functions.

Suppose that q^* is a local maximum or minimum of *u* on the set

$D = U \cap \{q : g_i(q) = 0,\ i = 1,..,k\}$ where $U \subset R^n$ is open.

Let $g(q) = \begin{pmatrix} g^1(q) \\ . \\ . \\ g^k(q) \end{pmatrix}$ and suppose also that $rank\left(Dg\left(q^*\right)\right) = k$.

Then there exists a vector (of Lagrangean multipliers) $\mu = \begin{pmatrix} \mu_1 \\ . \\ . \\ \mu_k \end{pmatrix} \in R^k$ such that

$$Du\left(q^*\right) + \sum_{i=1}^{k} \mu_i Dg^i\left(q^*\right) = 0 \qquad \text{(L)}$$

In this example, we have three constraints:

$g^1(q) = I - p(q_1)q_1 - p(q_2)q_2$

$g^{k+1}(q_1, q_2) = q_k \qquad \text{for} \qquad k = 1,2$

Hence, we need three Lagrangean multipliers $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$.

The Lagrangean function is given:

$$\begin{aligned} L(q_1 q_2; \mu_1 \mu_2, \mu_3) &= u(q_1, q_2) + \sum_{k=1}^{3} \mu_k g^k(q_1, q_2) \\ &= \ln q_1 + \ln q_2 + \mu_1(I - p_1(q_1)q_1 - p_2(q_2)q_2) + \mu_2 q_1 + \mu_3 q_2 \end{aligned}$$

The first order conditions

Set (I)

$$\frac{\partial L}{\partial q_1} = 0 \Leftrightarrow$$

$$MU_{q_1} - \mu_1 \frac{\partial}{\partial q_1}\left(p_1(q_1)q_1\right) + \mu_2 = 0 \Leftrightarrow \qquad (1)$$

$$\frac{1}{q_1} - \mu_1\left(p_1(q_1) + p_1'(q_1)q_1\right) + \mu_2 = 0$$

$$\frac{\partial L}{\partial q_2} = 0 \Leftrightarrow$$

$$MU_{q_2} - \mu_2 \frac{\partial}{\partial q_2}\left(p_2(q_2)q_2\right) + \mu_2 = 0 \Leftrightarrow \qquad (2)$$

$$\frac{1}{q_2} - \mu_1\left(p_2(q_2) + p_2'(q_2)q_2\right) + \mu_3 = 0$$

Set (II)

$$\frac{\partial L}{\partial \mu_1} = 0 \Leftrightarrow p_1(q_1)q_1 + p_2(q_2)q_2 = I \qquad (3)$$

$$\begin{array}{lcl} \dfrac{\partial L}{\partial \mu_2} \geq 0 & & q_2 \geq 0 \\ \mu_2 \geq 0 & \text{or} & \mu_2 \geq 0 \\ \dfrac{\partial L}{\partial \mu_2}\mu_2 = 0 & & q_2\mu_2 = 0 \end{array} \qquad (4)$$

$$\begin{array}{lcl} \dfrac{\partial L}{\partial \mu_3} \geq 0 & & q_3 \geq 0 \\ \mu_3 \geq 0 & \text{or} & \mu_3 \geq 0 \\ \dfrac{\partial L}{\partial \mu_3}\mu_3 = 0 & & q_3\mu_3 = 0 \end{array} \qquad (5)$$

Note that the Lagrange theorem gives only *necessary* conditions for the existence of a maximum. In other words, we claim that, if a (possibly local) maximum exists, it must satisfy equations (1)-(5).

For the required conditions of the Lagrangean theorem to apply:

(1) The condition $rank\left(Dg\left(q^*\right)\right)=k$ is called the *constraint qualification condition*. It plays a central role in the proof of the Lagrange theorem (in particular, it ensures that the Jacobian matrix $\underset{k\times n}{Dg\left(q^*\right)}$ contains an invertible $k\times k$ sub-matrix, which may be used to define the vector of Lagrangean multipliers μ). This is obvious by considering the fist-order condition (L) in matrix form:

$$L_q\left(q^*;\ \mu^*\right)+g_q\left(\pi^*\right)\mu^*=0\Leftrightarrow g_q\left(\pi^*\right)\mu^*=-L_q\left(q^*;\ \mu^*\right)$$

For a vector of Lagrange multipliers μ^* to exist such that (L) holds, there must exist some $k\times k$ sub-matrix of $\underset{n\times k}{g_q\left(q^*\right)}$ which is invertible.

More importantly, it turns out that if the constraint qualification condition is violated, then the conclusion of the theorem (i.e. equation (L)) may also fail. That is, if q^* is a local maximum at which $rank\left(Dg\left(q^*\right)\right)<k$, then there need not exist a vector $\mu\in R^k$ such that $Du\left(q^*\right)+\sum_{i=1}^{k}\mu_i Dg^i\left(q^*\right)=0$.

In this example, we have

$$Dg(q)=\begin{bmatrix}\frac{\partial g^1}{\partial q_1} & \frac{\partial g^2}{\partial q_1} & \frac{\partial g^3}{\partial q_1}\\ \frac{\partial g^1}{\partial q_2} & \frac{\partial g^2}{\partial q_2} & \frac{\partial g^3}{\partial q_2}\end{bmatrix}=\begin{bmatrix}p_1'(q_1)q_1+p_1(q_1) & 1 & 0\\ p_2'(q_2)q_2+p_2(q_2) & 0 & 1\end{bmatrix}$$

This is a *2x3* matrix and, therefore, $rank(Dg(q))\le 2$. Moreover, since it contains the non-singular sub-matrix $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, we have $rank(Dg(q))=2$ and the condition is satisfied at any point *q* (including, of course q^*).

(2) Both functions $u, g : R^2 \to R$ are C^1

(3) Since we do not allow the agent to consume negative amounts of any of the two commodities, $U = R^2_+$ which is an open subset of R^2

c. For sufficiency, we need to consider the second-order conditions for the problems of optimization under equality constraints. Recall our notation regarding the Langrangean function:

$$L(q_1 q_2 ; \mu_1 \mu_2, \mu_3) = u(q_1, q_2) + \sum_{k=1}^{3} \mu_k g^k (q_1, q_2)$$

We will also assume in this part that both u, g^i : $i = 1,..,k$ are C^2 functions.

Theorem:

Suppose there exist points $q^* \in D$, $\mu^* \in R^k$ such that:

1. $rank\left(Dg\left(q^*\right)\right) = k$ and
2. $Du\left(q^*\right) + \sum_{i=1}^{k} \mu_i^* Dg^i\left(q^*\right) = 0$

Define $Z\left(q^*\right) = \left\{z \in R^n : Dg\left(q^*\right)z = 0\right\}$ and let $D^2L\left(q\ ;\mu\right)$ denote the $n \times n$ matrix of the second derivative of $L\left(\cdot\ ;\mu\right)$ with respect to *q*: $D^2L\left(q\ ;\mu\right) = D^2u\left(q\right) + \sum_{i=1}^{k} \mu_i D^2 g^i\left(q\right)$.

Denote by D^2L^* the $n \times n$ matrix $D^2L\left(q^*;\mu^*\right) = D^2u\left(q^*\right) + \sum_{i=1}^{k} \mu_i^* D^2 g^i\left(q^*\right)$.

Then:

- If $z^T D^2 L^* z < 0$ for all $z \in Z\left(q^*\right)$ with $z \neq 0$, then q^* is a strict local maximum of *u* on *D*.
- If $z^T D^2 L^* z > 0$ for all $z \in Z\left(q^*\right)$ with $z \neq 0$, then q^* is a strict local minimum of *u* on *D*.

Note that the sufficient condition for local maximum (minimum) corresponds to the quadratic form $z^T D^2 L^* z$ being negative (positive) definite subject to the constraint $Dg\left(q^*\right)$ and, thus, to the symmetric matrix $D^2 L^*$ (which is the Hessian matrix of the Lagrangean function evaluated at the critical point$\left(q^*;\mu^*\right)$) being negative (positive) definite subject to the constraint $Dg\left(q^*\right)$. In other words, the Lagrangean function must be negative (positive) definite subject to the constraint $Dg\left(q^*\right)$, in the neighborhood of the critical point $\left(q^*,\mu^*\right)$.

To be able to make use of this theorem, we need to recall our notation from PS1/Question 2. In that question we saw that the definiteness of a symmetric $n\times n$ matrix *A* can be completely characterized in terms of its sub-matrices. We will now examine a related question: the characterization of the definiteness of *A* on only the set $\left[z\neq 0 : Bz = 0\right]$ where *B* is a $k\times n$ matrix of rank *k*.

Let $\pi = \left(\pi_1,\pi_2,..,\pi_n\right)$ a permutation of the integers $\{1,2,...,n\}$ and Π the set of all permutations of the integers $\{1,2,...,n\}$. Denote by A^π the symmetric *nxn* matrix obtained by applying the permutation π to both the rows and the columns of *A*:

$$A^\pi = \begin{bmatrix} a_{\pi_1\pi_1} & a_{\pi_1\pi_2} & . & a_{\pi_1\pi_n} \\ . & & & . \\ . & & & . \\ a_{\pi_n\pi_1} & a_{\pi_n\pi_2} & . & a_{\pi_n\pi_n} \end{bmatrix}$$

Let also B_k^π denote the $k\times n$ matrix obtained by applying the permutation π to only the columns of *B*:

$$B_k^\pi = \begin{bmatrix} b_{1\pi_1} & b_{1\pi_2} & . & b_{1\pi_n} \\ . & & & . \\ . & & & . \\ b_{k\pi_1} & b_{k\pi_2} & . & b_{k\pi_n} \end{bmatrix}$$

In an obvious extension of this notation, A_l^{π} will be the $l \times l$ sub-matrix obtained from A^{π} by retaining only the first l rows and l columns of A^{π}. Similarly, B_{kl}^{π} will denote the $k \times l$ sub-matrix obtained from B_k^{π} by retaining only the first l columns of B_k^{π}.

Denote also by A_l the sub-matrix $A_l = \begin{bmatrix} a_{11} & a_{12} & . & a_{1l} \\ . & & & . \\ . & & & . \\ a_{l1} & a_{l2} & . & a_{ll} \end{bmatrix}$. Similarly, denote by B_{kl} the sub-matrix $B_{kl} = \begin{bmatrix} b_{11} & b_{12} & . & b_{1l} \\ . & & & . \\ . & & & . \\ b_{k1} & b_{k2} & . & b_{kl} \end{bmatrix}$.

Finally, given any $l \in \{1,..,n\}$, let C_l the $(k+l)\times(k+l)$ matrix obtained by "bordering" the sub-matrix A_l by the sub-matrix B_{kl} in the following manner:

$$C_l = \begin{bmatrix} 0_k & B_{kl} \\ B_{kl}^T & A_l \end{bmatrix} = \begin{bmatrix} 0 & . & 0 & b_{11} & . & b_{1l} \\ . & & & & & \\ 0 & . & 0 & b_{k1} & & b_{kl} \\ b_{11} & & b_{k1} & a_{11} & & a_{1l} \\ . & & & & & \\ b_{1l} & . & b_{kl} & a_{l1} & & a_{ll} \end{bmatrix}$$

Theorem*[2,3]*:

Let A be a symmetric $n \times n$ matrix and B a $k \times n$ matrix such that $|B_k| \neq 0$. Then,

1. $z^T Az \geq 0$ for all z such that $Bz = 0$ if and only if and only if $(-1)^k |C_r^\pi| \geq 0$ for all $r \in (k+1,..,n)$ and for all $\pi \in \Pi$.
2. $z^T Az \leq 0$ for all z such that $Bz = 0$ if and only if and only if $(-1)^r |C_r^\pi| \geq 0$ for all $r \in (k+1,..,n)$ and for all $\pi \in \Pi$.
3. $z^T Az > 0$ for every z such that $Bz = 0$ if and only if and only if $(-1)^k |C_r| > 0$ for all $r \in (k+1,..,n)$.
4. $z^T Az < 0$ for every z such that $Bz = 0$ if and only if and only if $(-1)^r |C_r| > 0$ for all $r \in (k+1,..,n)$.

In the Lagrangean applications we need to consider two different cases for applying the theorem above:

1. When ***all*** of our constraints in the vector $g(q)$ are ***equality constraints*** then we take: $A = D^2L^*$ and $B = Dg(q^*)$. The matrices C_r are called the ***bordered Hessians*** since they are constructed by bordering an $r \times r$ sub-matrix of the Hessian D^2L^*, with the terms obtained from the matrix $Dg(q^*)$.
2. When we also have ***inequality constraints*** within the vector $g(q)$, then for the examination of the second order conditions we only consider those constraints that are actually binding at the critical point q^* under study. Let $g^B(q^*)$ the set of binding constraints at q^*. Now take: $A = D^2L^*$ but $B = Dg^B(q^*)$

By inspection of the objective function of our example, we see that neither of the non-negativity constraints $q_1 \geq 0,\ q_2 \geq 0$ can be binding at an optimal point. Hence, $k = 1$ and r can only take the value 2.

[2] Note the difference between parts 1 and 3 on the one hand and parts 2 and 4 on the other. In parts 1 and 3, the term (-1) is raised to the fixed power *k*, so that the signs of the determinants $|C_r^\pi|$ and $|C_r|$ are required to be all the same. In parts 2 and 4, this term is raised to the power *r*, so that the signs of these determinants must alternate.

[3] Note also that this theorem essentially claims that the definiteness of the matrix *A* subject to the constraint *B* is given by the definiteness of the bordered matrix *C*.

Hence, we have:

$$C_2 = \begin{bmatrix} 0 & \dfrac{\partial g^1}{\partial q_1} & \dfrac{\partial g^1}{\partial q_2} \\ \dfrac{\partial g^1}{\partial q_1} & \dfrac{\partial^2 L}{\partial q_1^2} & \dfrac{\partial^2 L}{\partial q_1 q_2} \\ \dfrac{\partial g^1}{\partial q_2} & \dfrac{\partial^2 L}{\partial q_2 q_1} & \dfrac{\partial^2 L}{\partial q_2^2} \end{bmatrix}$$

$$|C_2| = -\frac{\partial g^1}{\partial q_1}\begin{vmatrix} \dfrac{\partial g^1}{\partial q_1} & \dfrac{\partial^2 L}{\partial q_1 q_2} \\ \dfrac{\partial g^1}{\partial q_2} & \dfrac{\partial^2 L}{\partial q_2^2} \end{vmatrix} + \frac{\partial g^1}{\partial q_2}\begin{vmatrix} \dfrac{\partial g^1}{\partial q_1} & \dfrac{\partial^2 L}{\partial q_1^2} \\ \dfrac{\partial g^1}{\partial q_2} & \dfrac{\partial^2 L}{\partial q_2 q_1} \end{vmatrix}$$

$$= \left(p_1'(q_1)q_1 + p_1(q_1)\right)\begin{vmatrix} -\left(p_1'(q_1)q_1 + p_1(q_1)\right) & 0 \\ -\left(p_2'(q_2)q_2 + p_2(q_2)\right) & -\dfrac{1}{q_2^2} - \left(p_2''(q_2)q_2 + 2p_2'(q_2)\right) \end{vmatrix}$$

$$-\left(p_2'(q_2)q_2 + p_2(q_2)\right)\begin{vmatrix} -\left(p_1'(q_1)q_1 + p_1(q_1)\right) & -\dfrac{1}{q_1^2} - \left(p_1''(q_1)q_1 + 2p_1'(q_1)\right) \\ -\left(p_2'(q_2)q_2 + p_2(q_2)\right) & 0 \end{vmatrix}$$

$$= \left(p_1'(q_1)q_1 + p_1(q_1)\right)^2\left(\frac{1}{q_2^2} + p_2''(q_2)q_2 + 2p_2'(q_2)\right)$$

$$+ \left(p_2'(q_2)q_2 + p_2(q_2)\right)^2\left(\frac{1}{q_1^2} + p_1''(q_1)q_1 + 2p_1'(q_1)\right)$$

Note that:

$$|C_2| = \left(p_1'(q_1)q_1 + p_1(q_1)\right)^2\left(\frac{1}{q_2^2} + p_2''(q_2)q_2 + 2p_2'(q_2)\right)$$

$$+ \left(p_2'(q_2)q_2 + p_2(q_2)\right)^2\left(\frac{1}{q_1^2} + p_1''(q_1)q_1 + 2p_1'(q_1)\right) > 0 \Leftrightarrow$$

$$\left(\frac{1}{q_1^2} + p_1''(q_1)q_1 + 2p_1'(q_1)\right) < -\frac{\left(p_1'(q_1)q_1 + p_1(q_1)\right)^2}{\left(p_2'(q_2)q_2 + p_2(q_2)\right)^2}\left(\frac{1}{q_2^2} + p_2''(q_2)q_2 + 2p_2'(q_2)\right)$$

Since, $k=1$, $r=2$, if we had $|C_2|>0$ at the critical point q^*, this would be sufficient to give $z^T D^2 L^* z<0$ for every z such that $Dg\left(q^*\right)z=0$. Consequently, q^* would be a strict local maximum.

Clearly, this condition would be satisfied if we had (evaluated at $\left(q_1^*, q_2^*\right)$):

$$\frac{1}{q_2^2}+p_2''(q_2)q_2+2p_2'(q_2)>0 \quad \wedge \quad \frac{1}{q_1^2}+p_1''(q_1)q_1+2p_1'(q_1)>0 \qquad \text{(SOC)}$$

d.

Solving

$$\frac{1}{q_1}-\mu_1\left(\sqrt{q_1}+\frac{\sqrt{q_1}}{2}\right)+\mu_2=0 \qquad (1)$$

$$\frac{1}{q_2}-\mu_1\left(\sqrt{q_2}+\frac{\sqrt{q_2}}{2}\right)+\mu_3=0 \qquad (2)$$

$$\sqrt{q_1^3}+\sqrt{q_2^3}=I \qquad (3)$$

$$q_2 \geq 0$$
$$\mu_2 \geq 0 \qquad (4)$$
$$q_2\mu_2=0$$

$$q_3 \geq 0$$
$$\mu_3 \geq 0 \qquad (5)$$
$$q_3\mu_3=0$$

From the utility function we see that, at the optimal point, we must have: $q_1^*, q_2^* \neq 0$

(4), (5): $\mu_2^*=\mu_3^*=0$

Equations (1) and (2) simplify now to:

$$\frac{1}{q_1}-\mu_1\left(\sqrt{q_1}+\frac{\sqrt{q_1}}{2}\right)=0 \qquad (1.i)$$

$$\frac{1}{q_2}-\mu_1\left(\sqrt{q_2}+\frac{\sqrt{q_2}}{2}\right)=0 \qquad (2.i)$$

From either equation, it is obvious that: $\mu_1^* \neq 0$

Hence:

$$\frac{(1.i)}{(2.i)} \overset{\mu_1 \neq 0}{\Rightarrow} \frac{q_2}{q_1} = \frac{\sqrt{q_1} + \frac{\sqrt{q_1}}{2}}{\sqrt{q_2} + \frac{\sqrt{q_2}}{2}} = \frac{\sqrt{q_1}}{\sqrt{q_2}} \Leftrightarrow \sqrt{q_1^3} = \sqrt{q_2^3} \Leftrightarrow q_1 = q_2$$

We have: $2\sqrt{q_1^3} = I \Leftrightarrow q_1 = \sqrt[3]{\frac{I^2}{4}}$ And $q_2 = \sqrt[3]{\frac{I^2}{4}}$

We know that the required conditions of part (b) hold, in this setting, at all values of q.

For the conditions of part (c), you should verify that (SOC) holds indeed at the (LCP) $\left(q_1^*, q_2^*\right) = \left(\sqrt[3]{\frac{I^2}{4}}, \sqrt[3]{\frac{I^2}{4}}\right)$

6. The consumer's optimization problem:

$$\max_{x_1, x_2, x_3} u(x_1, x_2, x_3) = x_1^2 + \min\{x_2, x_3\}$$
$$s.t.$$
$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I$$
$$x_i \geq 0 \quad i = 1,2,3$$

The Weierstrass theorem cannot apply in this case because the utility (objective) function is not continuous on its domain $D = R_+^3 \cap \{(x_1, x_2, x_3) \in R^3 : p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I\}$.

Consequently, the Lagrange theorem does not apply either because the utility function is not C^1.