C103, Fall 20003, Problem Set 4 (due October 2)

1. Suppose a consumer has a concave utility function U(x), where x = $(x_1,...,x_n)$ is a vector of n goods and services, and maximizes utility subject to $x \geq 0$ and a budget constraint $p \cdot x \leq y$, where $p >> 0$ is a vector of strictly positive prices, and $y > 0$ is income. Assume that U(x) is increasing in x; i.e., if $x' << x''$, then $U(x') < U(x'')$. Let X(p,y) denote a vector-valued function, termed the *market* or *Marshallian* demand function, that maximizes U(x) subject to the budget constraint, and H(p,u) denote a vector-valued function, termed the *Hicksian* demand fuction, that minimizes $y = p \cdot x$ subject to the constraint that $U(x) \geq u$. The *indirect utility function* is defined as

$$V(p,y) \equiv U(X(p,y)) = \max\{U(x) \mid x \geq 0 \text{ and } p \cdot x \leq y\}$$

and the *expenditure function* is defined as

$$M(p,u) \equiv p \cdot H(p,u) = \min\{y \mid V(p,y) \geq u\} = \min\{p \cdot x \mid U(x) \geq u\}.$$

Suppose $x = (x_1,x_2,x_3)$ and the specific utility function $u = (x_1 \cdot x_2)^{1/3} + x_3$.

(A),Find the market demand functions $X_i(p,y)$ for i = 1,2,3, the Hicksian demand functions H(p,u), the indirect utility function, and the expenditure function. Verify the identities $X(p,y) \equiv H(p,v(p,y))$, $H(p,u) \equiv X(p,m(p,u))$, $y \equiv m(p,v(p,y))$, and $u \equiv v(p,m(p,u))$.

(B) *Roy's identity* says that $X_i(p,y) = -(\partial V(p,y)/\partial p_i)/(\partial V(p,y/ \partial y)$, provided the derivatives exist. Verify this identity for the given utility function.

(C) *Shephard's identity* says that $H_i(p,u) = \partial M(p,u)/\partial p_i$, provided the derivatives exist Verify this identity for the given utility function.

(D) Duality says that $U(x) = \min_p V(p,p \cdot x)$ and $U(x) = \min\{u \mid M(p,u) \geq p \cdot x \text{ for all } p \geq 0\}$. Verify these relations for the given utility function.

(D) Duality says that M(p,u) is a concave, linear homogeneous function of p, and V(p,y) is a convex, homogeneous of degree zero function of (y,p). Verify these conditions for the given utility function.

(E) Implications of the concavity of the expenditure function in p are

$$\partial H_i(p,u)/\partial p_j = \partial^2 M(p,u)/\partial p_i \partial p_j = \partial^2 M(p,u)/\partial p_j \partial p_i = \partial H_j(p,u)/\partial p_i,$$

so that the cross-price derivatives of Hicksian demands are symmetric, and

$$\partial H_i(p,u)/\partial p_i = \partial^2 M(p,u)/\partial p_i^2 \leq 0,$$

so that Hicksian demands are always downward sloping in their own prices. Verify these results for the given utility function.

(F) Use the identity $X(p,y) \equiv H(p,V(p,y))$ to conclude that

$$\partial X_i(p,y)/\partial p_j = (\partial H_i(p,u)/\partial p_j)|_{u=V(p,y)} + (\partial H_i(p,u)/\partial u)|_{u=v(p,y)} \cdot (\partial v(p,y)/\partial p_i)$$
$$\partial X_i(p,y)/\partial y = (\partial H_i(p,u)/\partial u)|_{u=V(p,y)} \cdot (\partial v(p,y)/\partial y).$$

Use the second equation to solve for $(\partial H_i(p,u)/\partial u)|_{u=v(p,y)}$ and substitute this into the first equation to obtain

$$\partial X_i(p,y)/\partial p_j = (\partial H_i(p,u)/\partial p_j)|_{u=V(p,y)} + ((\partial X_i(p,y)/\partial y)/(\partial V(p,y)/\partial p_j))\cdot(\partial V(p,y)/\partial p_i).$$

Regroup the last terms and use Roy's identity to conclude that

$$\partial X_i(p,y)/\partial p_j = (\partial H_i(p,u)/\partial p_j)|_{u=V(p,y)} - X_j(p,y)\cdot\partial X_i(p,y)/\partial y.$$

Finally, use Shephard's identity to rewrite this as

$$\partial X_i(p,y)/\partial p_j = (\partial^2 M_i(p,u)/\partial p_i \partial p_j)|_{u=V(p,y)} - X_j(p,y)\cdot\partial X_i(p,y)/\partial y.$$

This is termed the *Slutsky equation*, which shows that the effect of price p_j on the demand for good i can be decomposed into two terms, the first of which (termed the substitution effect) is symmetric in i and j and is negative for i = j, and the second of which (termed the income effect) is the product of the demand for good j and the derivative of the demand for good i with respect to income y. Good j is termed a *normal good* if its market demand is non-decreasing in income, and an *inferior good* when its demand is decreasing in income. The effect on the demand for good i of its own price will always be negative if good i is normal, as the income effect reinforces the substitution effect. However, if good i is inferior, the substitution and income effects are of opposite signs, and it is possible, but very unusual, for the income effect to outweigh the income effect so that the demand for good i rises when its price rises. Verify the Slutsky equation for the specific utility function above, and show that all the goods are normal, so all demand functions are downward sloping in their own prices.

2. Suppose a firm has a production possibility set **T** containing all the feasible net output vectors z (i.e., components of x are negative for inputs, positive for outputs). The firm seeks to maximize profit $\pi = p\cdot z$ subject to the constraint of staying within the production possibility set, where $p >> 0$ is a vector of prices. If a profit maximum is attainable, let Z(p) denote the *net supply function* giving a net output vector that attains it, and let $\pi = \Pi(p) \equiv p\cdot Z(p) = \max_{z\in \mathbf{T}} p\cdot z$ denote the *profit function* giving the level of profit attained. The profit function is convex and homogeneous of degree one in p, and its derivative satisfies $\partial\Pi(p)/\partial p_i = Z_i(p)$ when it exists.

(A) For the specific production possibility set $\mathbf{T} = \{(z_1,z_2,z_3)\in\mathbb{R}^3 \mid z_2 \le 0, z_3 \le 0, z_1 \le (z_2\cdot z_3^3)^{1/5}\}$, find Z(p) and $\Pi(p)$. Verify that $\Pi(p)$ is convex, with $\partial\Pi(p)/\partial p_i = Z_i(p)$.

(B) If **T** is a convex, closed set with the free disposal property that whenever it contains a vector, it also contains all vectors to the southwest of it, then $\mathbf{T} = \{z \in\mathbb{R}^3 \mid p\cdot z \le \Pi(p) \text{ for all } p >> 0\}$. Use this property to derive the production possibility set **T** associated with the profit function $\Pi(p) = (p_1^2 + 2p_2^2)/p_3$, and complete the circle by showing that $\Pi(p)$ is the profit function for this production possibility set.