Problem Set #7

Suggested Solutions

1. Consider that John is playing (O,F) according to the mixed strategy $(p,1-p): p\in[0,1]$ whereas Mary is playing according to the mixed strategy $(q,1-q): q\in[0,1]$.

The expected payoff to John is given:

$$E\pi_1(p,q) =$$
$$p(2q+0(1-q))+(1-p)(q+3(1-q))$$
$$=3(1-p)+4pq-2q$$

John seeks to choose his optimal strategy (i.e. p) in order to maximize this quantity. Differentiating with respect to p, we get:

$$\frac{\partial E\pi_1(p,q)}{\partial p} = -3+4q$$

Notice that

- For $q \le \frac{3}{4}$, we get $\frac{\partial E\pi_1(p,q)}{\partial p} \le 0$. Given that Mary playing a mixed strategy that assigns probability no greater than 0.75 to Opera, John's expected payoff is decreasing in the probability p that he also assigns to playing Opera. Therefore, his optimal response would be $p=0$ (i.e. playing the pure strategy Football).
- For $q > \frac{3}{4}$, we get $\frac{\partial E\pi_1(p,q)}{\partial p} > 0$. Given that Mary is playing Opera with probability at least 0.75, John's expected payoff is now increasing in the probability p that he assigns to playing Opera. His optimal response would be $p=1$ (i.e. playing the pure strategy Opera).
- For $q = \frac{3}{4}$, we get $\frac{\partial E\pi_1(p,q)}{\partial p} = 0$. If Mary is playing Opera with exactly 0.75 probability, his expected payoff is independent of the probability p. Here, any $p\in[0,1]$ is optimal (i.e. playing any of his two pure strategies Football, Opera as well as any mixture between them).

Similarly, Mary's expected payoff is given:

$$E\pi_2(p,q) =$$
$$q(3p+(1-p))+(1-q)(0p+2(1-p))$$
$$=(2-q)(1-p)+3pq$$

Mary seeks to choose her optimal strategy (i.e. q) in order to maximize this quantity. Differentiating with respect to q, we get:

$$\frac{\partial E\pi_2(p,q)}{\partial q} = 3p-(1-p) = 4p-1$$

Notice that

- For $p \leq \frac{1}{4}$, we get $\frac{\partial E\pi_2(p,q)}{\partial q} \leq 0$. Mary's expected payoff is decreasing in the probability q that she assigns to playing Opera, when John is playing Opera with probability no greater than 0.25. Therefore, her optimal response would be $q = 0$ (i.e. playing the pure strategy Football).
- For $p > \frac{1}{4}$, we get $\frac{\partial E\pi_2(p,q)}{\partial q} > 0$. When John is playing Opera with a probability greater than 0.25, her optimal response would be $q = 1$ (i.e. playing the pure strategy Opera).
- For $p = \frac{1}{4}$, we get $\frac{\partial E\pi_2(p,q)}{\partial q} = 0$. If John is playing Opera with probability of exactly 0.25, Her expected payoff is now independent of the probability q. Here, any $q \in [0,1]$ is optimal.

Putting the best responses of the two players together we have the diagram of Fig. I (note that the solid line depicts Mary's best response function while the dotted one depicts that of John):

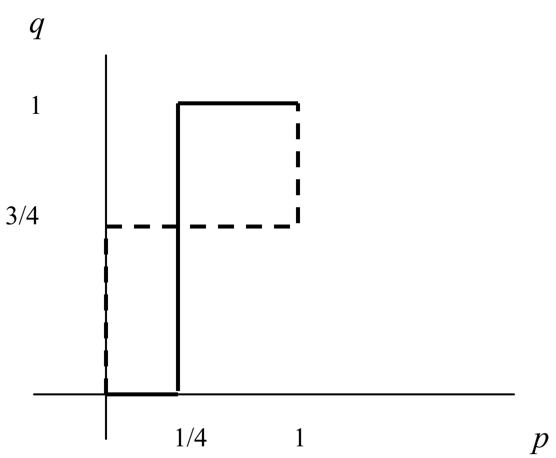

Fig. I

We can see, therefore, that we have three NE in this game: the two pure-strategy NE $\{(O,O),(F,F)\}$ and the mixed-strategy NE $\left(\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{3}{4},\frac{1}{4}\right)\right)$ where John plays Opera with probability 0.25 while Mary plays Opera with probability 0.75.

2. Notice that, in this variation of the game, Opera is a strictly dominant strategy for Mary. Individual rationality on her part dictates, therefore, that she chooses Opera irrespectively of John's play.

Assuming that John knows that Mary is rational (first-stage knowledge of rationality), John can deduce that Mary will play nothing but Opera. Hence, John would be play his best response against Opera, which Opera.
This game has a unique NE: (O,O)

3. Consider the strategic decision problem of player 1 against player 2 playing some action $z\in[0,1]$.
His optimization problem is given: $\max\limits_x f(x,z)$
Since *f* is concave in *x* given *z*, for any given value of *z*, this problem will have a solution. This solution defines a correspondence between the strategy *z* of player 2 and player 1's optimal response. This is, therefore, the best response correspondence of player 1:
$BR_1(z)=\arg\max\limits_x f(x,z)$
Note also that, since *f* is continuous, the argmax function will also be continuous in *z*.

Similarly, for the optimization problem of player 2, since *g* is also concave in *z* given *x*, the best response correspondence of player 2 is also well-defined:
$BR_2(x)=\arg\max\limits_z g(x,z)$
Since *g* is continuous, the best response correspondence of player 2 will also be continuous in *x*.

We know that a NE is a pair of strategies (x^*,z^*) such that each player is playing her best response against the NE strategy of the other.
Define the following correspondence: $BR:(x,z)\to(BR_1(z),BR_2(x))$.
This is:

1. Continuous in (x,z), on its domain $[0,1]^2$, because the component correspondences $BR_1(.),BR_2(.)$ are continuous.
2. Defined on the convex and compact set $[0,1]^2$.
3. Convex valued. This is easily seen as a result of the concavity of the functions *f*,*g* and the defining property of the argmax correspondences.
   Notice, for example, that if $x',x''\in\arg\max\limits_x f(x,z)$, we have:
   $$f(x',z)=f(x'',z)\geq f(\tilde{x},z)\ \ \forall\tilde{x}\in domf(x,z).$$

However, the same is clearly true for the convex combination $\lambda x' + (1-\lambda)x''$:

$$f\left(\lambda x' + (1-\lambda)x'', z\right) \geq \lambda f(x', z) + (1-\lambda) f(x'', z) \geq f(\tilde{x}, z) \quad \forall \tilde{x} \in domf(x,z).$$

By Kakutani's fixed point theorem, there exists a $\left(x^*, z^*\right)$ such that $\left(x^*, z^*\right) \in \left(BR_1\left(z^*\right), BR_2\left(x^*\right)\right)$. Such a strategy vector is a NE.

Note that, had the question actually assumed strict concavity for the payoff functions *f*,*g*, then the best responses would be uniquely defined strategies *x*,*z* for players 1 and 2 respectively. In other words, $BR_1(z) = \arg\max_x f(x,z)$ and $BR_2(x) = \arg\max_z g(x,z)$ would actually be functions and so would our construction *BR*. In that case, we could use Brower's fixed point theorem, which does not require verification of step 3 above.

4. Let S_i denote the set of pure strategies available to player *i*, $\sigma = (\sigma_1, ..., \sigma_I)$ denote a NE mixed strategy profile for the *I* players in a game and $S_i^+ \subset S_i$ denote the set of pure strategies that player *i* assigns positive probability to playing in his equilibrium mixed strategy σ_i.

The given assertion is that the expected payoff to player I from any pure strategy in the support of his NE mixed strategy σ_i is the same. In other words:

$$u_i\left(s_i, \sigma_{-i}\right) = u_i\left(s_i', \sigma_{-i}\right) \quad \forall s_i, s_i' \in S_i^+ \qquad \text{(I)}$$

To see the validity of the claim suppose that it were not true. Then we ought to be able to find strategies $s_i, s_i' \in S_i^+$ such that

$$u_i\left(s_i, \sigma_{-i}\right) > u_i\left(s_i', \sigma_{-i}\right).$$

If this was the case, however, player *i* could strictly increase his payoff by playing strategy s_i whenever his mixed strategy σ_i instructs him to play s_i'. In other words, shifting probability mass away from s_i' onto s_i improves upon the expected payoff of σ_i itself. But this is a contradiction for σ_i is supposed to be a NE strategy and, thus, there can be no other pure or mixed strategy (i.e. no re-allocation of probability weights amongst the pure strategies in S_i) that does better than σ_i against σ_{-i}.

The claim that all probability mixtures in the support S_i^+ of σ_i have the same expected payoff against σ_{-i} (equal, obviously, to the payoff $u_i\left(\sigma_i, \sigma_{-i}\right)$ of σ_i itself) is an immediate consequently of condition (I).

If $\quad u_i\left(s_i, \sigma_{-i}\right) = u_i\left(s_i', \sigma_{-i}\right) = c \quad \forall s_i, s_i' \in S_i^+$,

then clearly

$$\sum_{s_i \in S_i^+} p_i\left(s_i\right) u_i\left(s_i, \sigma_{-i}\right) = c \sum_{s_i \in S_i^+} p_i\left(s_i\right) = c = u_i\left(\sigma_i, \sigma_{-i}\right)$$

Therefore, any probability mixture $\left(p_i\left(s_i\right)\right)_{s_i \in S_i^+}$ over the support S_i^+ of σ_i (where $p_i\left(s_i\right)$ denotes the probability assigned to the pure strategy $s_i \in S_i^+$), gives the same expected payoff against σ_{-i} as σ_i itself.

5. Clearly, since every firm has zero costs and seeks to maximize profits, no firm will charge a negative price – i.e. $p, r \geq 0$. Moreover, no firm will choose a price that is strictly above 1 for the following reason:

- Even without considering the competition from the right firm, the left firm L, for example, can only attract customers if: $1-p-x \geq 0 \Leftrightarrow x \leq 1-p$
  Hence, for any $p \geq 1$, demand and, thus, profits are zero.
- Similarly, the right firm R faces the same problem: $1-r-(1-x) \geq 0 \Leftrightarrow r \leq x$.
  Since $x \in [0,1]$, firm R cannot charge a price greater or equal to 1 either.

Hence, both firms must follow pricing policies that lie strictly within the unit interval $p, r \in (0,1)$.

Notice, moreover, that once both firms are pricing within the unit interval, the indirect utility that a customer, at any location $x \in [0,1]$, gets from buying from either firm is strictly positive. Thus, for $p, r \in (0,1)$, consumers at all locations will definitely buy from one of the firms as they strictly prefer this to not buying at all. Consequently, we need not worry about the middle zero term in the consumers' indirect utility optimization objective $\{1-p-x, 0, 1-r-(1-x)\}$.

Having this in mind, let us examine the decision problem of the left firm given that the right firm charges a price $r \in (0,1)$.

The left firm attracts customers from locations x such that:

$$1-p-x \geq 1-r-(1-x) \Leftrightarrow x \leq \frac{1+r-p}{2} = x_0$$

Its profits, therefore, will be given:

$$\pi_L = p x_0 = p\frac{1+r-p}{2} = \frac{1}{2}\left(p + rp - p^2\right)$$

The firm's decision problem is:

$$\max_p \ \frac{1}{2}\left(p + rp - p^2\right)$$

The first order condition: $1+r-2p=0 \Leftrightarrow p = \dfrac{1+r}{2}$

The best response function, therefore, of this firm when the right one charges a price of r is: $BR_L(r) = \dfrac{1+r}{2}$ (I)

Similarly, the right firm attracts customers from locations *x* such that:

$$1 - p - x \leq 1 - r - (1 - x) \Leftrightarrow x \geq \frac{1 + r - p}{2} = x_0$$

Its profits, therefore, will be given:

$$\pi_R = r(1 - x_0) = r\left(1 - \frac{1 + r - p}{2}\right) = \frac{1}{2}\left(r + rp - r^2\right)$$

The firm's decision problem is:

$$\max_r \ \frac{1}{2}\left(r + rp - r^2\right)$$

This is qualitatively identical to the optimization problem of firm 1. Hence, the best response function of the right firm when the left one charges a price of *p* is:

$$BR_R(p) = \frac{1 + p}{2} \qquad \text{(II)}$$

For a NE, we have to solve the system of equations (I),(II) simultaneously.
This gives, however, the solution $p = r = 1$, which is clearly not acceptable[1]. Hence, this game, as given, does not have a NE.

6. This question is more or less an application of our analysis in question 3. The best response correspondences for the two players are here:

$BR_1(z) = \arg\max_x f(x, z)$ and $BR_2(x) = \arg\max_z f(z, x)$

Since *f* is concave and continuous in both *x* and *z*, these best responses are well-defined and continuous. We know that a NE is a pair of strategies (x^*, z^*) such that each player is playing her best response against the NE strategy of the other. Hence, if one defines again the correspondence: $BR : (x, z) \rightarrow (BR_1(z), BR_2(x))$, gives at the same conclusion as in problem 3 –namely that a NE exists.
However, in this problem we are interested in deriving a symmetric NE –i.e. one where both players play the same strategy. Let this equilibrium strategy be x^*.
Clearly, x^* must solve:

$$x^* = BR_1(x^*) = \arg\max_x f(x, x^*) \qquad (1)$$

and

$$x^* = BR_2(x^*) = \arg\max_z f(x^*, z) \qquad (2)$$

Conditions (1) and (2), however, are equivalent to:

$$f(x^*, x^*) \geq f(x, x^*) \quad \forall x \qquad (1.\text{i})$$

and

$$f(x^*, x^*) \geq f(x^*, x) \quad \forall x \qquad (2.\text{i})$$

[1] Recall that, by charging a price of 1, each firm gets no customers because it offers customers of all location negative indirect utility and the customers prefer to abstain from buying from any of the two firms.

Due to the symmetry of the payoff function *f* in its arguments:

$$f(x, x^*) = f(x^*, x) \quad \forall x.$$

Therefore, the two conditions are equivalent to one another and finding a symmetric NE reduces to finding a point x^* such that:

$$x^* = \arg\max_x f(x, x^*) \Leftrightarrow x^* = BR_1(x^*)$$

This is nothing but a fixed point for the best response correspondence BR_1. Its existence is guaranteed by Kakutani's fixed point theorem, since the best response is continuous on its domain, defined on a compact set (by assumption) and convex valued[2].

For an example of a symmetric game where only asymmetric pure equilibria exist consider the game below:

	O	F
O	-1,-1	0,0
F	0,0	-1,-1

This is clearly a symmetric game with only two pure NE: one where the row player plays F while the column player plays O and one where the column player plays F and the row player is responding optimally by choosing O. Both of these equilibria are asymmetric in that the two players are playing different equilibrium strategies.

Note, however, that even this example *does* have a symmetric NE -although it does so in mixed strategies. Each player playing O with probability 0.5 is a NE (and a symmetric one since both players are using the same probability mixture over their pure strategies). This is expected, of course, since, allowing for mixed strategies, turns the choice space of the payoff function from the discrete, two point set $\{O, F\}$ into the convex and compact unit simplex in R^2 $\{(p,q) : p,q \in [0,1]\}$. Moreover, as convex combinations of the two pure strategy payoff values -1 and 0, the payoff functions are continuous in *p*,*q*. Hence, our result regarding the existence of a symmetric NE applies.

[2] See problem 3 for the relevant argument here. Note, however, that we really need the domain of BR_1 to be a convex set in order to apply Kakutani's theorem. This is not really given by the question but it should have been so.