Chapter V.2

FACTOR SUBSTITUTION IN THE INTERINDUSTRY MODEL AND THE USE OF INCONSISTENT AGGREGATION*

PETTER FRENGER

Central Bureau of Statistics of Norway

1. Introduction

Do input prices affect the observed variability of the input–output coefficients? Classical theory of production tells us they ought to, while Leontief claims they don't make any difference since substitution is infeasible, though the essence of his claim is an empirical one [Leontief (1951, p. 40)]: "The assumption of fixed coefficient of production necessarily entails the existence of some disparity between our theoretical scheme and the actual industrial setup it is intended to represent. Empirical investigation alone can reveal how significant the disparity actually is."

A very extensive literature has developed, investigating the stability of these coefficients, and the best summary of the state of the arts is probably still the one given by Barna (1963, p. 6): "It is by now generally recognized that input–output coefficients are not constant. But there is a great deal of stability in these coefficients and the practical question is to what extent and with what modifications the input–output model may be used in empirical applications."

Some researchers have tried, by various methods, to test the Leontief Hypothesis, starting with Leontief (1951) himself. Others have looked for other factors, mainly aggregation errors and changing industry tech-

*I wish to thank D. McFadden for valuable assistance at various stages of this work.

nology, to help explain the observed variations. The role of relative factor prices, however, has seldom been studied.

Cameron (1952) finds that: "There is significantly little evidence of substitution relations between inputs, and still less of diminishing technical marginal rates of substitution" and "evidence of price substitution is surprisingly meager." He does not tell, however, how he reached these conclusions.

In their thorough analysis, Arrow and Hoffenberg (1959, p. 43) ignored prices, giving the following explanation:

> "From the usual viewpoint of economic theory, the omission of relative prices as explanatory variables for change in input–output coefficients is very conspicuous. The hypothesis is indeed made that in the short-run methods of production cannot respond to price variations. The basic reason for making this assumption is simplicity, i.e., the gain in degrees of freedom obtained by omitting some variables."

In the introduction to the 1954 Norwegian input–output tables, Sevaldson (1960, p. 53) writes:

> "Studies done in connection with the preparation of part II of this publication (i.e., with the empirical preparation of the input–output table) show that for almost all industries, the input coefficients are more influenced by changes in the product mix than by input substitution. The conclusion would be that substitution only exceptionally plays a significant role, with the degree of precision offered by a 100–150 sector table. Lack of sector homogeneity makes product mix the dominant source of changes in the coefficients. There is, however, one sort of substitution which cannot be ignored, i.e. the substitution among domestic and imported goods."

In another study, Sevaldson (1963) talks about theoretically possible price substitution, but he does not find any empirical evidence of a price effect.

In her extensive study on the changing structure of the U.S. economy, Ann Carter (1970, p. 13) writes: "Input structures of individual industries are apparently sensitive to changes in relative prices in the inputs." In only a few places does she document these findings, and in these cases the price changes seem to be entirely technologically induced during the 11 years between the two tables she is comparing (1947 and 1958).

There is one large body of literature concerning the effect of prices on input coefficients: the whole theory and estimation of neoclassical production functions on all levels of aggregation. Unfortunately this literature has generally limited itself to the primary factors of production, usually capital and labor, ignoring the intermediate inputs. The evidence seems to indicate that the substitution among capital and labor is significant, and there is no *a priori* reason to believe that substitution should be limited to these two inputs.

The first attempts at testing the applicability of the Generalized Leontief model were done by Diewert on the Canadian labor market, allowing for substitution between different kinds of labor and capital (Diewert (1969a), and on primary inputs in the aggregate U.S. economy (Diewert 1969b), both excluding the intermediate inputs.

There is one other body of literature relevant to the present analysis. While Leontief's theory is generally formulated in physical terms and tested in constant prices, Klein (1952–1953) has shown that stability of the value coefficients is implied by a Cobb–Douglas production function, which, of course, admits substitution among all the inputs. Tests of the relative stability of the input coefficients expressed in constant and current prices have been carried out by Haldi, Sevaldson, and Tilanus.

In a study of the U.S. boot and shoe industry between 1919 and 1954, Haldi (1959) found the value coefficient significantly more stable in the case of labor, while it was insignificantly so in the case of materials' input. Sevaldson (1963) finds on the whole the materials' input (cork) more stable in volume terms, while labor appears more stable in value terms in his study of the Norwegian cork industry.

The Dutch input–output tables were originally available in current prices only, and most of Tilanus' (1966) experiments were performed on these tables. But he did also derive sectoral price deflators, and compared the predictions achieved by the two methods. He found mean square value prediction errors one year ahead smaller than the volume prediction errors in 18 out of 27 sectors, and concluded that: "the classical input–output assumption that volume coefficients are constants is less workable than the hypothesis ... that value coefficients are constant."[1] And "medianwise, mean square volume prediction errors are some 15% larger than mean square value prediction errors."[2]

2. The Model

We will rely on the Generalized Leontief (G.L.) cost function [Diewert (1971)] to analyze and test the price responsiveness of the input–output

[1]Tilanus (1966, p. 80). As he states, his results might be partly due to the unsatisfactory quality of some of the price indices he is forced to use.

[2]Tilanus (1966).

coefficients. Assuming constant returns to scale, the function is

$$C(y,\mathbf{p}) = y \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij}(p_i p_j)^{1/2}, \tag{1}$$

where $\mathbf{p}$ is the input price vector, y is output, and $b_{ij} = b_{ji}$. The linearity of the function in the unknown coefficients makes it well suited for empirical investigation, and it further reduces to the Leontief fixed coefficient cost function when $b_{ij} = 0$, $i \neq j$.

Differentiating equation (1) with respect to p_i and dividing by the output rate, we obtain from Shephard's Lemma (1953)

$$\frac{x_i(y,\mathbf{p})}{y} = \sum_{j=1}^{m} b_{ij} \left(\frac{p_j}{p_i}\right)^{1/2}, \qquad i = 1,\dots,m, \tag{2}$$

where $x_i(y,\mathbf{p})$ are the factor demand equations. The set of equations (2) express the input–output coefficients as functions of the input prices.[3] The b_{ij}, $i \neq j$, coefficients in equation (1) are generally required to be non-negative since this is a sufficient condition for the cost function to be concave in prices. Several of the estimated b_{ij} coefficients, however, turn out to be negative. This would indicate a complementary relationship between the ith and the jth input, because the demand for the ith input would then become a decreasing function of the price of the jth input, i.e.,

$$\frac{\partial x_i(y,\mathbf{p})}{\partial p_j} = \frac{1}{2}\, y b_{ij}(p_i p_j)^{-1/2} < 0. \tag{3}$$

With negative b_{ij} coefficients, it becomes necessary to test whether the estimated functions are concave in a reasonable neighborhood of the observed prices. Diewert (1971, pp. 501–503) gives a detailed discussion of these conditions. It is cumbersome to carry out the full test,[4] and I have limited myself to the following two partial tests. Concavity of equation (1) requires that the Hessian matrix be negative semi-definite,

[3]The G.L. production function was formulated with an exponent equal one half, and this formulation is used below. The function could however be generalized to the form

$$C(y,p) = y \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} p_i^{\alpha} p_j^{1-\alpha},$$

with $\alpha \in [0,1]$ (even further generalization is possible). Linearity in the coefficients is lost if α is to be estimated from the data, while it is difficult to obtain *a priori* estimates of this coefficient. Some experiments with different values of α showed the coefficients b_{ij} to be very sensitive to significant changes in α ($\alpha \sim .1$), while the sum of squared residuals varied very little.

[4]For a derivation and analysis of the complete test see Lau, Appendix A.4 in Volume 1.

and the first-order principal minor of this Hessian is just

$$\frac{\partial^2 C(y,\mathbf{p})}{\partial p_i^2} = -\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{m} b_{ij} p_j^{1/2} p_i^{-3/2} \leq 0, \qquad i = 1,\ldots,m, \tag{4}$$

which must be non-positive.

The second test is provided by the estimated shadow elasticities of substitution (SES). The SES between the *i*th and the *j*th inputs can be written

$$\begin{aligned} \text{SES}_{ij} &= \frac{-C_{ii}/x_i^2 + 2(C_{ij}/x_i x_j) - C_{jj}/x_j^2}{1/p_i x_i + 1/p_j x_j} \\ &= \frac{1/x_i^2 x_j^2}{1/p_i x_i + 1/p_j x_j} \begin{vmatrix} 0 & x_i & x_j \\ x_i & C_{ii} & C_{ij} \\ x_j & C_{ij} & C_{jj} \end{vmatrix} \geq 0, \end{aligned} \tag{5}$$

where the determinant on the right-hand side is the lowest-order principal minor of the bordered Hessian (i.e., subject to the constraint that total cost remain constant) of equation (1). If the cost function is concave, this determinant and the SES must be positive.[5]

3. The Data

The data for this study have all been taken from the two volumes "National Accounts, Classified by Fourteen and Five Industrial Sectors, 1949–1961" published by the Central Bureau of Statistics of Norway in 1965 (Vol. I) and 1966 (Vol. II). The Bureau obtained this data by aggregation from substantially more detailed tables.

These publications contain 13 annual input–output tables for the Norwegian economy, presented in constant (1955) prices, together with the price indices for each interindustry transaction for each year. The second volume presents supplementary data on capital and labor inputs. The tables have 14 industrial sectors with inputs analogously subdivided, but further distinguished as to whether they are domestic or imported.[6] As explained below, three of these sectors were chosen for the following study.[7]

[5]See McFadden (1963, p. 74, footnote 4).

[6]For a study of the behavior of the input–output coefficients derived from these and the more disaggregated tables, see Sevaldson (1970).

[7]1949, the first year of the period being studied, is generally regarded as representing the beginning of a four-year period of normalization of the economy, after the post-war reconstruction effort.

The data, as presented in the publications, are all in buyers' prices. But the theory of production tells us that the cost of bringing the output to the consumer ought not to affect the cost minimizing input decisions of the producer. It would, of course, enter into his profit calculations. Deliveries from sector 55 ("trade, transport, and communication") include the gross trade margins of the various industries. These trade margins, which in many cases exhausted the deliveries from sector 55, were therefore not considered as an input in the production process. Except for this correction, and some aggregation of the input sectors, the data for the intermediate inputs were used as they appear in the tables.

3.1. *Labor Input*

In all the input–output tables, primary inputs are aggregated into value-added, and its volume and price index is defined residually given the volume and value of the intermediate inputs and the output. Labor input is presented separately in man-years both for wage-earners and the self-employed, and so is the current wage rate. For the labor input, I used both wage-earners and the self-employed, imputing to the latter the average industry wage. That such a procedure is not always justified is illustrated by the sector agriculture (not estimated below), for which such an imputation more than exhausts the value added.

3.2. *Capital Input*

The input of capital has been assumed proportional to the capital stock at the beginning of the year. This would seem reasonable since the period was characterized by a steady expansion of output (GNP declined only in 1968, and then by 1.8% in volume terms), and in the three industries considered below, net investment was negative only for one industry (textiles) in one year (1958). I am further implicitly assuming that the input of capital services in any one year is optimal given the prices and the output level. This assumption can probably best be defended as a first approximation. A superior alternative would perhaps be to take the capital stock as given, and minimize cost subject to the available capital stock, or to introduce a more realistic investment theory.

If one believed that static perfect competition prevailed, one could compute the price of capital services (c) residually so as to preserve the equality

$$q\cdot y = C(y;\mathbf{p},w,c) = \sum_{i=1}^{n} p_i\cdot x_i(y;\mathbf{p},w,c) + w\cdot L(y;\mathbf{p},w,c) + c\cdot K(y;\mathbf{p},w,c), \tag{6}$$

where $\mathbf{p}$ is the vector of the intermediate input prices, q the output price, w the wage rate, y the output, and x_i, L, and K the cost minimizing intermediate, labor, and capital inputs, respectively. This "rate of return" method has been used in some of the computations below. I have at other times broken the left-hand equality in equation (6), and determined the cost of capital by the user cost expression,[8]

$$c(t) = p_I(t)(r(t) + d(t)), \tag{7}$$

where $p_I(t)$ is the cost of capital goods in period t, $r(t)$ a rate of interest, and $d(t)$ the rate of depreciation in period t. Norwegian statistics on the rate of interest are, particularly for the first part of the period, somewhat incomplete. To obtain a series extending over the whole period, I have spliced together two different series[9] which were overlapping in the years 1954 and 1955.

4. Estimation

The cross equation constraints imposed by the condition that the coefficient matrix $\mathbf{B}$ be symmetric [see equation (1)] necessitate the simultaneous estimation of the whole system of n equations. If one feels it warranted to believe that $\mathbf{\Sigma}\otimes\mathbf{I}_T = \sigma^2\mathbf{I}_{nT}$, ordinary least square becomes the appropriate estimation method, and gives rise to the coefficient estimate,[10]

$$\hat{\mathbf{b}} = (\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{a}, \tag{8}$$

[8]See Jorgensen (1967). I have ignored the capital gains term, and corporate income tax considerations.

[9]The two series are (i) weighted average on current yield of various 2.5% government bonds for the years 1949–1955 (IMF 1966/67), and (ii) the average lending rate for all commercial banks for the years 1954 to 1961 [Norway (1954–61)], the latter series determining the level of the combined series.

[10]This method was used for some early estimates primarily for ease of computation. The estimated coefficients are fairly sensitive to the method of estimation.

where $\mathbf{P}$ is an $(nT)\times\frac{1}{2}n(n+1)$ matrix of the square roots of relative prices. But since the input–output coefficients may vary by a factor of 10 for different inputs, it is not a plausible assumption. A study of the residuals also tends to indicate that the error terms for a given year are significantly correlated. Generalized least squares is called for, and since the contemporaneous covariance matrix is unknown, Zellner's (1962) two-stage procedure was utilized. The estimate $\hat{\Sigma}$ of Σ was obtained by ordinary least squares regression on the individual equation (2), without the imposition of symmetry constrained and then estimating the vectors $\hat{\mathbf{e}}_i$ of residuals. Forming the matrix $\hat{\mathbf{E}} = (\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\ldots,\hat{\mathbf{e}}_n)$, we get the following unbiased estimate of the covariance matrix:

$$\hat{\Sigma} = \frac{1}{T-n}\hat{\mathbf{E}}'\hat{\mathbf{E}},$$

and the two-stage coefficient estimate becomes

$$\mathbf{b}^* = (\mathbf{P}'(\hat{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{P})^{-1}\mathbf{P}'(\hat{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{a}. \tag{9}$$

If $\hat{\Sigma}$ tends in probability to Σ as T tends to infinity, the estimator $\mathbf{b}^*$ will be asymptotically equivalent to the Aitken estimator (i.e., assuming Σ were known).[11] Thus $\mathbf{b}^*$ is a consistent estimator of $\mathbf{b}$, and Kakwani (1967) has shown that $\mathbf{b}^*$ will also be unbiased if the error terms are symmetrically distributed.

Hypothesis Testing

We will frequently below test the validity of restrictions imposed upon the coefficients of the $\mathbf{B}$ matrix. By application of the Chow (1960) test, this would be a straightforward task if the errors were normally distributed and their covariance matrix $\mathbf{\Omega} = \Sigma\otimes\mathbf{I}_T$ were known. Let $Q_0 = \mathbf{e}_0'\mathbf{\Omega}^{-1}\mathbf{e}_0$ be the weighted sum of squares residuals resulting from the estimation of n factor demand equations involving k coefficients, and let $Q_1 = \mathbf{e}_1'\mathbf{\Omega}^{-1}\mathbf{e}_1$ be the weighted sum of squares residuals after the imposition of q constraints. Then, under the assumption that the constraints are true,

$$F = \frac{Q_1 - Q_0}{Q_0}\frac{nT-k}{q} \tag{10}$$

[11]See Malinvaud (1966, pp. 286–290). The variance of the estimate $\mathbf{b}^*$ is approximated by

$$[\mathbf{P}'(\hat{\Sigma}^{-1}\otimes\mathbf{I}_T)\mathbf{P}]^{-1}.$$

to an order of smallness of $1/T$ [Zellner (1962, p. 352)].

will have an F distribution with q and $nT - k$ degrees of freedom.

We do not know the distribution of the error terms, including $\mathbf{\Omega}$, and are thus forced to rely, somewhat unsatisfactorily given the small size of our sample, on asymptotic properties. Zellner (1962) again suggests replacing $\mathbf{\Omega}$ by its consistent estimator $\hat{\mathbf{\Omega}}$. The resulting statistic will have an asymptotic F distribution, if the constraint is true.

5. Empirical Results

Three industrial sectors were chosen for the present study, and for each G.L. factor demand equations were estimated. The three sectors are: textiles, construction and metals.[12]

Textiles. This sector was chosen because it seemed to be a relatively homogeneous sector producing essentially consumer goods. The textile output declined in three of the years in the sample, while net investment was negative in one. The capital input and its price were only computed using the "rate of return" method [see equation (6)]. Two sets of G.L. production functions were computed. In one case aggregating to domestically produced and imported inputs, plus capital and labor, and in the other, to six exhaustive sectors of intermediate inputs.

Construction. This was the first sector to be studied because, at the time, it seemed to be the only sector providing all the necessary data. Its output was strongly regulated during the period under study, and declined in three of those years. In this case only the "user cost" of capital [see equation (7)] was used. The level of capital stock was very low in the beginning of the period, and the net rate of return on capital varied from 178% to 14%. The intermediate inputs were aggregated according to the five-sector tables, and over imported and domestically produced inputs, giving four intermediate and two primary input sectors.

Metals. This industry produces almost entirely goods for export, and was thus believed to reflect competitive behavior more strongly than the other two more protected industries. It includes the output of most metals, ores, fertilizers and carbides, and of the total output of 2589 mill.kr. in 1961, 18% were intermediate deliveries, and 78% were exported. The output increased uniformly over the period. Only the four largest intermediate inputs were included in the estimation of the G.L. factor demand equations, thus assuming the inputs of the smaller sectors

[12]Detailed results are presented in Section 10.1, together with an explanation of the classification system.

to be used in fixed proportions. Both the "rate of return" and the "user cost" of capital were used.

The estimated results, and particularly the estimates of the individual b_{ij} coefficients, are of somewhat mixed quality. This is probably in part an identification problem. Many of the off-diagonal coefficients are negative, indicating an apparent complementary relationship among the relevant inputs. It is unlikely that complementarity is so prevalent at this high level of aggregation: its appearance might indicate the presence of other factors which the model, in its present form, does not account for.

The expected evidence for substitution among domestic and imported inputs, where this distinction was maintained, did not materialize. It is inconclusive for textiles (*xD* versus *xF*, Appendix – Table A.1, and *x14* versus *x64*, Table A.2; see Section 10), and contradicted in the case of metals (*x14* versus *x64*, Table A.6). The level of aggregation and a significant difference in the composition of goods within these supposedly equivalent categories, may well explain this inconsistency. Labor and capital, on the other hand, do show in all cases the expected substitution relationship.

The estimated cost functions were all strictly increasing in the input prices. For 1955, the base year of the price system with $p_i = 1.00$, this condition which is equivalent to the non-negativity of the estimated demand equations, becomes

$$\frac{x_i(y,\mathbf{p})}{y} = \sum_{j=1}^{n} b_{ij} \geq 0, \qquad i = 1,\ldots,n. \tag{11}$$

The expression is the fitted coefficient value for that year. It also presents a measure of the relative importance of the input, and is presented along with the estimated symmetric G.L. functions in Tables A.1, A.2, and A.4 through A.6 in Section 10.

Are the estimated factor demand equations derived from a cost function which is concave in the prices? Section 2 gave two necessary conditions of which the first [see equation (4)] in the base year becomes equivalent to

$$\sum_{j \neq i} b_{ij} \geq 0, \qquad i = 1,\ldots,n. \tag{12}$$

This condition is violated in about one out of six cases.[13]

A further test of concavity is provided by the estimated shadow

[13]These off-diagonal sums are presented in the symmetric G.L. tables (Tables A.1, A.2 and A.4 through A.6 in Section 10). The variance of the sum was estimated in a few, mainly "negative" cases, and never found significantly different from zero.

elasticities of substitution, since the numerator of the SES is just a quadratic form of the Hessian of the cost function, subject to the constraint that cost is constant [see equation (5)]. The SES for textiles, construction, and metals for the year 1955 are presented in Table 1. Out of the 36 elasticities of substitution, four turn out to be negative, and three of these occur in the construction industry.

TABLE 1
1955 shadow elasticities of substitution.

(a) Textiles

	xD	*xF*	*xK*	*xL*
xD		1.1990	1.0416	1.4620
xF			0.0754	0.6413
xK				0.7197
xL				

xD and *xF* are domestically produced and imported intermediate inputs, respectively. *xK* is capital and *xL* is labor.
Source: Table A.1.

(b) Construction

	x1	*x2*	*x3*	*x5*	*xK*	*xL*
x1		0.0691	0.1781	0.2345	1.1943	1.1573
x2			0.5636	−0.1527	1.0216	−0.3711
x3				−0.1323	0.8766	0.0804
x5					0.5704	0.0345
xK						1.3689
xL						

x1, *x2*, *x3*, *x4* are intermediate inputs of consumer goods, capital goods, export goods, and services, respectively. All six "intermediate" shadow elasticities of substitution were positive when only the four intermediate inputs were included in the estimation.
Source: Table A.4.

(c) Metals

	x14	*x53*	*x64*	*x83*	*xK*	*xL*
x14		0.2560	−0.9392	0.9779	0.8991	1.1122
x53			1.8630	0.8432	1.1465	1.4307
x64				2.2139	1.5620	1.6008
x83					0.7384	1.2251
xK						0.7467
xL						

x14, *x53*, *x64*, and *x83* are intermediate inputs of "other consumer goods", "electricity", "imported other consumer goods", and "imported metals", respectively.
Source: Table A.6.

5.1. *The Symmetry Condition*

The existence of a production function implies an ability on the part of the producer to allocate his resources in a technically efficient manner. The existence of a cost function implies in addition that production is so structured as to produce a given output at the lowest possible cost. Such a cost function will imply a symmetry constraint in the response of the producer to changes in the factor prices.

In the case of the G.L. cost function, these constraints are satisfied if the **B** matrix is symmetric, i.e., $b_{ij} = b_{ji}$ for all i and j. Thus, in order to test for cost-minimizing behavior, we have to test for the validity of the condition $b_{ij} = b_{ji}$, which is a simple parametric restriction on the coefficients of the unconstrained factor demand equations [see equation (10)].[14]

The results presented in Table 2 are inconclusive, but it is clear that

TABLE 2
F tests of the symmetry condition.[a]

		Textiles	Construction	Metals
		6 intermediate inputs	4 intermediate & 2 primary inputs	4 intermediate & 2 primary inputs
Intermediate inputs only	Incl. primary inputs as fixed coefficient sectors	$F(15,66) = 0.05$ $F^* = 1.82/2.33$	$F(6,60) = 5.4$ $F^* = 2.25/3.12$	$F(6,60) = 1.4$ $F^* = 2.25/3.12$
	Not incl. primary inputs	$F(15,42) = 0.56$ $F^* = 1.92/2.50$	$F(6,36) = 18.4$ $F^* = 2.36/3.35$	$F(6,36) = 2.5$ $F^* = 2.36/3.35$
		2 intermediate & 2 primary inputs		
Intermediate and primary inputs	Estimation using "user cost"		$F(15,42) = 2.3$ $F^* = 1.92/2.50$	$F(15,42) = 0.96$ $F^* = 1.92/2.50$
	Estimation using "rate of return"	$F(6,36) = 5.4$ $F^* = 2.36/3.35$		$F(15,42) = 2.72$ $F^* = 1.92/2.50$

[a]F^* is the critical value of the F statistic at the 5% and 1% confidence level, respectively.

[14]The matrix $\mathbf{B} = (b_{ij})$, when introduced in the definition of the G.L. function, need not be symmetric, but only the sum $b_{ij} + b_{ji}$ would then be identified. When estimating the derived factor demand equations, however, the symmetry condition is necessary, as otherwise these demand functions would not be integrable into a cost function.

rejection of the symmetry condition does not, without further investigation, permit rejection of cost minimization for the industries studied. Some or all of the following conditions might also contribute to the rejection of symmetry:

(1) The G.L. production function is an inappropriate specification, and it might be that the true cost function is best approximated by a non-symmetric G.L. function.
(2) The data have been aggregated over firms, commodities, and capital goods, which might lead to significant distortions.
(3) The price indices used, particularly for capital, might be faulty.
(4) The static character of the analysis might be, and probably is inappropriate, particularly, perhaps, its stipulation of an always optimal level of capital.

Textiles. The symmetry condition is rejected ($F = 5.4$) when aggregated to the level of four inputs: domestic intermediate, imported intermediate, capital, and labor. Visual examination of the estimated coefficients indicate that all inputs fare about equally badly, with perhaps capital relatively worse. When intermediate inputs are disaggregated, the picture changes ($F = 0.56$) despite (because of?) the fact that primary inputs are excluded with, one would expect, important price variables. This might suggest that rejection was partly due to aggregation: much of the variability in the component price variables was in this case lost in the aggregation.

Construction. Symmetry is again rejected at the 5%, but not at the 1% level. It is more clearly rejected when only the intermediate inputs are included.[15]

Metals. This sector gave the "best" results: $F = 2.7$ using the rate of return cost of capital, and $F = 0.96$ using the rate of interest. The main difference seems to be the greater ability of the rate of return cost of capital in the unconstrained version to "explain" the input of imported metals.

One reason for the frequent rejection of symmetry seems to be that many of the price variables do become proxy variables for other excluded determinants of coefficient variability when the symmetry constraint is not imposed. Several price variables move rather uniformly

[15]In the analysis I have always taken the input prices as exogenous and dealt separately with each sector of the economy, where prices and output will generally be determined simultaneously. Only the import prices would seem to be legitimate exogenous variables. It is hard to evaluate this simultaneous equation effect on the results.

and could act as time trends. The imposition of symmetry tends to reduce such errant behavior by the price variable.

It would, in concluding, tentatively seem that disaggregation tends to improve the results, that the exclusion of certain sectors tends to worsen them, and, perhaps, that some important variables are still excluded.

5.2. *The Leontief Hypothesis*

Does this extension and complication, almost beyond recognition, of the simple Leontief model provide a better insight into the variability of interindustry flows? Table 3 presents a test of the Leontief hypothesis within the broader G.L. framework. By imposing the restriction that all the off-diagonal elements be zero, i.e., $b_{ij} = 0$ for all $i \neq j$, and applying the Chow test, we get the F statistics shown in the table.[16] I have tested the Leontief model against the G.L. model (and not against the unsymmetric factor demand equations) so as to remain within a system that can be derived from a cost function.

There would seem to be little doubt that the Leontief assumption would have to be rejected: when all the inputs are included, the F

TABLE 3
F tests of Generalized Leontief versus Leontief.[a]

		Textiles	Construction	Metals
Intermediate inputs only	Incl. primary inputs as fixed coefficient sectors		$F(6,66) = 4.1$ $F^* = 2.24/3.09$	$F(6,66) = 1.8$[b] $F^* = 2.24/3.09$
	Not incl. primary inputs		$F(6,42) = 5.6$ $F^* = 2.32/3.26$	$F(6,42) = 3.0$ $F^* = 2.32/3.26$
Intermediate and primary inputs	Estimation using "user cost"		$F(15,57) = 10.0$ $F^* = 1.85/2.38$	$F(15,57) = 12.4$ $F^* = 1.85/2.38$
	Estimation using "rate of return"	$F(6,42) = 14.7$ $F^* = 2.32/3.26$		$F(15,57) = 13.1$ $F^* = 1.85/2.38$

[a] F^* is the critical value of the F statistic at the 5% and 1% confidence level, respectively.
[b] Volume of capital input determined using the rate of return.

[16] When all the inputs are not included in the G.L. estimation, the question arises whether the omitted sectors ought to be ignored or included as fixed coefficients sectors in estimating the F statistics.

statistics are in all cases greater than 10. Only for metals, using intermediate inputs only, is there some doubt. If the last year of the sample is excluded, the F statistics change from 3.0 to 3.8 and from 1.8 to 2.0, the former now significant at the 1% level. The problem is partly caused by input of imported metals (*x83*), the input coefficient of which makes a drastic and unexplained drop in 1961. Also, the input coefficient for labor declines steadily over the period giving the labor input, in the Leontief case, a very high sum of squared residuals. When labor is included as a fixed coefficient sector in the estimation of the intermediate inputs G.L. function, it alone alters the F statistic by a factor of 0.5.

The existence of such trends in the input coefficients, trends which the price variables may help explain either as the true explanatory variables or as proxy variables, is one of the main reasons for the improved performance of the G.L. function. On the whole the greatest changes have occurred in the capital and labor coefficients, and this explains the great difference in the test statistics between the instances where these inputs were included or excluded.

6. 1961 Predictions

An alternative test of the Generalized Leontief model, and one of special interest for its potential applications, is its ability to predict the input–output coefficients of a given year. Table 4 summarizes some of the results of reestimating the G.L. functions for the period 1949–1960 and using the estimated **B** matrix to forecast the 1961 input–output coefficients.[17]

Traditional input–output forecasts require knowledge of the final demand vector, and so would in practice the G.L. method. In the following I am, however, only predicting the 1961 coefficients, and since the whole system is linear homogeneous, "only" a knowledge of the price vector of the year to be predicted is required.

Several different forecasting methods have been tested and compared:

(a) the "average coefficient" for the period 1949–1960;
(b) the "1960 coefficient";
(c) the "average coefficient with time trend", i.e., estimating the function $a_{ij}(t) = d_{ij} + d'_{ij} \cdot t$, $t =$ 1949, 1950,...,1960, and using the estimated values of d_{ij} and d'_{ij} to predict the 1961 value of a_{ij}.

[17]The detailed results are presented in Tables A.7–A.9 in Section 10.

The above methods are among the traditional input–output forecasting methods,[18] and we are primarily interested in comparing their performance with methods which allow for variations in the relative prices. The first two such methods presented below are:

(d) the "Generalized Leontief" function;
(e) the "unsymmetric Generalized Leontief" function, i.e., applying the G.L. function estimate without the imposition of the symmetry constraint.

The utilization of method (e) presents us with an *ad hoc* test of the effect of changing relative prices. These functions, however, cannot be derived from any underlying cost function. Tables A.7–A.9 in Section 10 present the actual 1961 input–output coefficients and the 1961 relative prediction errors (RPE) for each input, where

$$\mathrm{RPE} = \frac{\text{Prediction} - \text{Realization}}{\text{Realization}}.$$

The weighted average (weighted by the 1961 coefficients) of the absolute values of the relative prediction errors are presented in Table 4 as summary statistics. It is felt that the latter is a better measure of the seriousness of the forecasting errors, since it takes into account the relative size of the various input sectors.

TABLE 4
Summary of 1961 predictions;[a] weighted average of relative prediction errors.

Prediction method	Textiles[b]	Construction	Metals[c]
(a) *Traditional methods*			
1949–1960 average coefficient	0.1462	0.1325	0.2696
1949–1960 average coefficient with time trend	0.0553	0.0429	0.2470
1960 coefficient	0.0704	0.0579	0.2440
(b) *Generalized Leontief*			
Symmetric	0.1031	0.0259	0.2813
Unsymmetric	N.A.	0.0276	N.A.
(c) *G.L.: 1960–1961 coefficient change*			
Symmetric	0.0879	0.0163	0.2837
Unsymmetric	N.A.	0.0276	N.A.

[a]For more detail, see Tables A.7–A.9.
[b]Using two aggregate intermediate inputs (xD, xF), and capital and labor.
[c]Using the "user cost" of capital.

[18]See for example Tilanus (1966).

Looking first at the three traditional forecasting methods, it is seen that the average coefficient is clearly inferior to the other two methods. Among the latter the "average coefficient with trend" does better than the "1960 coefficient" in two of the three industries, while it is a tie in the third.[19]

Returning to the G.L. function we find a set of rather mixed results. The G.L. function does very well when applied to the construction industry, reducing considerably the weighted average RPE. Comparing the symmetric and the unsymmetric G.L. forecasts, the former does slightly better despite its inability to predict the increased level of capital input, contradicting to some extent the rejection of the symmetry condition at the 1% level. Applied to the textile industry, the G.L. function is inferior to the 1960 coefficient and the trend forecasts when four aggregated inputs are considered. The comparison improves somewhat when the inputs are disaggregated.

As for the metals industry, every method does poorly and the G.L. method does slightly worse than the three traditional methods. This industry presents a rather erratic picture in 1961, with the input of imported consumer goods, not elsewhere classified, (*x64*) increasing by 118% and the input of imported metals (*x83*) decreasing by 24% as compared with 1960.[20] The addition of the two primary sectors to the four intermediate sectors does, however, seem to improve upon the results.

As an alternative forecasting method one may use the estimated factor demand functions to predict the coefficient change from 1960 to 1961 rather than the 1961 coefficient level, i.e.,

$$a^{P}_{61} = {}_{GL}a^{P}_{61} - {}_{GL}a_{60} + a_{60}, \tag{13}$$

where a^{P}_{61} is the predicted 1961 coefficient using the present method, ${}_{GL}a^{P}_{61}$ is the G.L. predicted 1961 coefficient, ${}_{GL}a_{60}$ is the G.L. 1960 fitted value, and a_{60} is the actual 1960 coefficient.

[19]This is somewhat at variance with Tilanus (1966, Ch. 7), who generally finds the last available input coefficients to give superior results, when compared with trend extrapolations. Ten out of sixteen input sectors were in our case found to have statistically significant time trends. In the construction industry all but the small sector *x5* had significant trends.

[20]The direction of the price response is also contrary to the experience over the first twelve years, when the two inputs were estimated to be insignificantly complementary. Adding 1961 to the sample renders the two inputs significant substitutes. This changing relationship only worsens the 1961 G.L. predictions as compared with those methods that do not allow for price effects.

If the 1960 G.L. fitted value and the 1960 coefficient were identical, the present method would lead to the same results as presented above. They are likely to differ because of the random error, but also because of technological change and factor substitution. The observed 1960 coefficient would incorporate these changes up through that year, allowing the present method to partly incorporate these factors in predicting the 1961 coefficient. Further the 1960 coefficients might not be optimal due to lags in the adjustment of factor inputs, particularly that of capital. These lags are likely to persist and also affect the 1961 coefficient, suggesting again that theoretically the method of equation (13) may represent an improvement. Equation (13) can also be justified simply on the grounds that it incorporates into the 1961 forecast the most recent information on this variable.

Several estimates were performed, and they are again presented in Tables A.7–A.9 in Section 10. The method does not seem, generally, to lead to any improvement in the estimate, with the significant exception of the symmetric G.L. function applied to the construction industry. The weighted average of the RPE's is in the latter case reduced from 0.026 to 0.016, mainly because of a greater ability to account for the changing capital and labor coefficients. The G.L. function (both the symmetric and the unsymmetric version) is further able to predict correctly the direction of change of the coefficients, including the one turning point.

For the textile industry all three turning points are missed at the most aggregated level, while two out of four were wrongly predicted when estimating intermediate inputs only. The predicted direction of change was correct in those cases where the trend persisted. Again the improvement is predominantly in the capital and labor sectors.

The metal industry again presents the worst picture with about half the directions of change wrongly predicted. Every direction of change is wrongly predicted when estimating intermediate inputs only, suggesting again the significance of including also the primary inputs.

In conclusion, the results of the investigation are somewhat mixed, though they do seem to indicate that relative prices have a significant effect on the variability of the input–output coefficients. The standard Leontief model is rather clearly rejected. The test of the symmetry condition is inconclusive, and the forecasting results are better or equal to those achieved by the more traditional methods. There is also some evidence that the usual explanatory variables: "aggregation errors" and "technical change" are significant.

7. Inconsistent Shadow Elasticities of Substitution

I will, in the final three sections of this paper, analyze one specific problem of aggregation: the use of inconsistent price aggregates, that is price aggregates that explicitly or implicitly make the wrong assumptions about the substitutability of the disaggregated inputs. The usual Paasche price indices, which were used in the previous sections, tend to overestimate the true shadow elasticity of substitution by a factor of two.[21] The following argument will formalize the intuitive feeling that an overestimate of the substitutability at the lowest level may introduce an opposite bias in the substitution parameters at the aggregate level, and that the shadow elasticities of substitutions presented in Table 1 underestimate the true elasticities.[22]

Let the production structure be described by the cost function,

$$C = C(y;p_1,p_2,\ldots,p_n), \tag{14}$$

which is assumed to be weakly separable with respect to the partition $\{\mathbf{N}_i : i = 1,\ldots,r\}$ of the index set $\mathbf{N} = \{1,\ldots,n\}$.[23] It is further assumed that there exists linearly homogeneous consistent price aggregates ρ^i so that the cost function can be written as

$$C = \mathscr{C}[y;\rho^1(\mathbf{p}^1),\rho^2(\mathbf{p}^2),\ldots,\rho^r(\mathbf{p}^r)], \tag{15}$$

where $\mathbf{p}^v = (p_k : k \in N_v)$, $v = 1,\ldots,r$. The functions ρ^v will normally be unknown and in practice an inconsistent price aggregate π^v will be used. By so doing we are postulating the existence of an inconsistently aggregated cost function B defined implicitly by

$$C = B(y;\pi^1,\pi^2,\ldots,\pi^r), \tag{16}$$

where the value of C on the left-hand side is given by (14). The function B is a function not only of the π^v's (and the p_k's), but also of the change in these variables (i.e., $d\pi^v$ and dp_k). The partial derivative of B w.r.t. π^v is defined by

$$B_v = \frac{d_vC}{d\pi^v} \quad \text{where} \quad d_vC = \sum_{k\in \mathbf{N}_v} x_k \, dp_k.$$

[21]See equation (22) in Section 7. An even more extreme example of a "wrong" aggregation procedure is the Laspeyres quantity index with its implicit assumption of an infinite elasticity of substitution.

[22]This is probably not, however, the main reason that some of the SES's are negative.

[23]Separability is introduced in part to simplify the derivation, but more importantly to insure the existence of uniquely defined consistent aggregate elasticities of substitution by which the bias may be judged.

We are interested in analyzing the inconsistently aggregated shadow elasticities of substitution of B defined, in a manner analogous to the consistently aggregated SES, by

$$\mathrm{SES}_{ij}^{B} = -\frac{B_j}{B_i}\,\mathrm{d}\left(\frac{B_i}{B_j}\right)\Big/\frac{\pi^j}{\pi^i}\,\mathrm{d}\left(\frac{\pi^i}{\pi^j}\right) = -\frac{\mathrm{d}B_i/B_i - \mathrm{d}B_j/B_j}{\mathrm{d}\pi^i/\pi^i - \mathrm{d}\pi^j/\pi^j}, \tag{17}$$

holding total cost, output, and the price aggregates π^k, $k \neq i,j$, constant. The price changes are thus limited to movements along the aggregate factor price frontier with $B_i\,\mathrm{d}\pi^i = -B_j\,\mathrm{d}\pi^j$.

To concentrate the analysis on the consequences of misspecifying the second-order parameters of the aggregating functions, it will be assumed that these functions satisfy the first-order conditions for consistent aggregation for the point (called the base point) at which the analysis occurs, i.e., that

$$\frac{\partial \pi^i}{\partial p_k}\Big/\frac{\partial \pi^i}{\partial p_l} = \frac{\partial \rho^i}{\partial p_k}\Big/\frac{\partial \rho^i}{\partial p_l} = \frac{x_k}{x_l}, \qquad k,l \in \mathbf{N}_v. \tag{18}$$

This condition is for example satisfied, for small price changes around the base point, by the usual Paasche (current weighted) price index,

$$\pi_p^v = \sum_{k\in \mathbf{N}_v} p_k^1 x_k^1 \Big/ \sum_{k\in \mathbf{N}_v} p_k^0 x_k^1, \tag{19}$$

where 0 indicates the base period and 1 the current period. Let Σ_{ij} be the true SES between sectors i and j in the consistently aggregated cost function $\mathscr{C}(y;\rho^1,\ldots,\rho^r)$. Let σ_{kl} be the SES between sectors k and l in the same separable input set N_v (this will also be the SES between the same two inputs in the consistent price aggregate ρ^v), and let σ_{kl}^{π} be the (implicit) aggregation SES between the same two inputs for the inconsistent price aggregate π^v. Define

$$\alpha_k = p_k x_k \Big/ \sum_{s\in \mathbf{N}_v} p_s x_s,$$

$$P_{kl} = \frac{\mathrm{d}p_k/p_k - \mathrm{d}p_l/p_l}{\mathrm{d}_v C/C^v}, \qquad k,l \in \mathbf{N}_v,\ v = i,j,$$

$$C^v = \sum_{k\in \mathbf{N}_v} p_k x_k \quad \text{and} \quad \mathrm{d}_v C = \sum_{k\in \mathbf{N}_v} x_k\,\mathrm{d}p_k.$$

It can then be shown that[24]

$$\mathrm{SES}_{ij}^{B} = \Sigma_{ij} + \frac{(1/C^i)\Delta_i + (1/C^j)\Delta_j}{1/C^i + 1/C^j} = \Sigma_{ij} + \Delta_{ij}, \tag{20}$$

[24]See Frenger (1975, pp. 110–119). The derivation there is mainly in terms of the production function and the direct elasticity of substitution, but the two are mathematically dual.

where Δ_{ij} is defined by the right-hand equality and Δ_v, $v = i,j$, is given by

$$\Delta_v = \frac{1}{4}\sum_{k\in \mathbf{N}_v}\sum_{\substack{l\in \mathbf{N}_v\\ l\neq k}}\Big[(\alpha_k+\alpha_l)(\sigma_{kl}-\sigma^{\pi}_{kl}) - \alpha_l \sum_{\substack{s\in \mathbf{N}_v\\ s\neq k}}(\alpha_k+\alpha_s)(\sigma_{ks}-\sigma^{\pi}_{ks})$$

$$-\alpha_k \sum_{\substack{s\in \mathbf{N}_v\\ s\neq l}}(\alpha_l+\alpha_s)(\sigma_{ls}-\sigma^{\pi}_{ls})$$

$$+\alpha_k\alpha_l \sum_{s\in \mathbf{N}_v}\sum_{\substack{t\in \mathbf{N}_v\\ t\neq s}}(\alpha_s+\alpha_t)(\sigma_{st}-\sigma^{\pi}_{st})\Big] P^2_{kl}. \tag{21}$$

Δ_v is linearly homogeneous of degree 0 in prices and inputs. For small deviations from the base point, the inconsistently aggregated SES differs from the true SES by a factor which is a weighted average of the sectoral error terms Δ_v. These error terms are seen to be functions of the misspecification of the second-order parameters of the inconsistent price aggregates, i.e., of the difference $(\sigma_{kl} - \sigma^{\pi}_{kl})$, and of the difference between the relative price changes of the disaggregated inputs. There are two extreme cases under which the error terms Δ_v will vanish:

(1) If $\sigma_{kl} = \sigma^{\pi}_{kl}$ for every $k,l \in \mathbf{N}_v$, i.e., if consistent price aggregates have been used.
(2) If $\mathrm{d}p_k/p_k = \mathrm{d}p_l/p_l$ for every $k,l \in \mathbf{N}_v$, i.e., if all price changes within the sector are proportional. This is just Hicks' (1946, p. 33) condition for consistent aggregation.

The empirical part of this paper uses Paasche price aggregates. Frenger (1975, p. 93) shows that in this case the implicit (aggregation) SES σ^{π}_{kl} will be given by

$$\sigma^{\pi}_{kl} = 2\sigma_{kl}, \tag{22}$$

and the sectoral error terms become

$$\Delta_v = -\frac{1}{4}\sum_{k\in \mathbf{N}_v}\sum_{\substack{l\in \mathbf{N}_v\\ l\neq k}}\Big[(\alpha_k+\alpha_l)\sigma_{kl} - \alpha_l \sum_{\substack{s\in \mathbf{N}_v\\ s\neq k}}(\alpha_k+\alpha_s)\sigma_{ks}$$

$$-\alpha_k \sum_{\substack{s\in \mathbf{N}_v\\ s\neq l}}(\alpha_l+\alpha_s)\sigma_{ls} + \alpha_k\alpha_l \sum_{s\in \mathbf{N}_v}\sum_{\substack{t\in \mathbf{N}_v\\ t\neq s}}(\alpha_s+\alpha_t)\sigma_{st}\Big] P^2_{kl}$$

$$= -\frac{1}{4}\sum_{k\in \mathbf{N}_v}\sum_{\substack{l\in \mathbf{N}_v\\ l\neq k}}\Big[P^2_{kl} - \sum_{\substack{s\in \mathbf{N}_v\\ s\neq k}}\alpha_s P^2_{ks} - \sum_{\substack{s\in \mathbf{N}_v\\ s\neq l}}\alpha_s P^2_{ls}$$

$$+\sum_{s\in \mathbf{N}_v}\sum_{\substack{t\in \mathbf{N}_v\\ t\neq s}}\alpha_s\alpha_t P^2_{st}\Big](\alpha_k+\alpha_l)\sigma_{kl}, \qquad v = 1,\ldots,r. \tag{23}$$

If the consistent aggregates were of the CES variety or, somewhat weaker, if all $\sigma_{kl} = \sigma_v$ for $k,l \in \mathrm{N}_v$, $k \neq l$, at the base point, then the sectoral error terms reduce to

$$\Delta_v = -\frac{1}{2}\sigma_v \sum_{k\in \mathrm{N}_v} \sum_{\substack{l\in \mathrm{N}_v \\ l\neq k}} \alpha_k \alpha_l P_{kl}^2, \qquad v = 1,\ldots,r. \tag{24}$$

If further there are only two inputs in the ith and the jth input group, i.e., $\mathrm{N}_i = \{1,2\}$ and $\mathrm{N}_j = \{3,4\}$,[25] then the inconsistently aggregated SES reduces to

$$\mathrm{SES}_{ij}^B = \Sigma_{ij} - \left[\frac{\sigma_{12}\alpha_1\alpha_2}{C^i}\left(\frac{\mathrm{d}p_1/p_1 - \mathrm{d}p_2/p_2}{\mathrm{d}_i C/C^i}\right)^2 + \frac{\sigma_{34}\alpha_3\alpha_4}{C^j} \times \left(\frac{\mathrm{d}p_3/p_3 - \mathrm{d}p_4/p_4}{\mathrm{d}_j C/C^j}\right)^2\right]\left(\frac{1}{C^i} + \frac{1}{C^j}\right)^{-1}, \tag{25}$$

where

$$C^i = x_1 p_1 + x_2 p_2, \qquad C^j = x_3 p_3 + x_4 p_4,$$
$$\mathrm{d}_i C = x_1\,\mathrm{d}p_1 + x_2\,\mathrm{d}p_2, \qquad \mathrm{d}_j C = x_3\,\mathrm{d}p_3 + x_4\,\mathrm{d}p_4,$$

and

$$\mathrm{d}_i C = -\mathrm{d}_j C.$$

Following are two examples of what can happen when the price changes are not proportional. Let us first simplify (25) further by assuming that $p_1 = p_2 = p_3 = p_4 = 1$ at the base point, and that $p_1 x_1 = p_2 x_2 = p_3 x_3 = p_4 x_4$, i.e., that each input sector is of the same magnitude. This gives

$$\mathrm{SES}_{ij}^B = \Sigma_{ij} - \frac{1}{2}\left[\sigma_{12}\left(\frac{\mathrm{d}p_1 - \mathrm{d}p_2}{\mathrm{d}p_1 + \mathrm{d}p_2}\right)^2 + \sigma_{34}\left(\frac{\mathrm{d}p_3 - \mathrm{d}p_4}{\mathrm{d}p_3 + \mathrm{d}p_4}\right)^2\right]. \tag{26}$$

Example 1: Assume that only one input price changes, e.g., $\mathrm{d}p_1 > 0$ and $\mathrm{d}p_2 = \mathrm{d}p_3 = \mathrm{d}p_4 = 0$. We then first have to normalize this price change to insure that we remain on the factor price frontier. The normalized price changes will become $\mathrm{d}p_k' = -\frac{1}{3}\mathrm{d}p_1'$, $k = 2,3,4$. Since the prices in N_j change proportionately, Δ_j will vanish, and the inconsistently aggregated SES becomes

$$\mathrm{SES}_{ij}^B = \Sigma_{ij} - 2\sigma_{12}.$$

[25]This example does not require the consistent aggregate to be of the CES variety, but with only two inputs the CES price aggregate will give a second order approximation for any aggregate at the base point.

Example 2: Assume that $dp_2 = dp_4 = 0$ and that p_1 and p_3 change in opposite direction such that $dp_3 = -dp_1$. Then

$$\mathrm{SES}^B_{ij} = \Sigma_{ij} - \tfrac{1}{2}(\sigma_{12} + \sigma_{34}).$$

It is easy to see from either of these examples that the inconsistently aggregated SES could become negative for many plausible values of Σ_{ij} and the σ_{kl}'s. It should also be noted that as long as $\sigma_{kl} = \sigma_v$ for every $k,l \in \mathrm{N}_v$, then $\Delta_v \leq 0$ [see equation (24)], and the inconsistent method of estimation will underestimate the true value of Σ_{ij}. No such definite conclusion can be drawn when $\sigma_{kl} \neq \sigma_{st}$ for some $k,l,s,t \in \mathrm{N}_v$, because there will then exist price changes for which Δ_v would be positive, but Δ_v will be negative for most price changes.

If one knows, or is prepared to make sufficient assumptions about the disaggregated elasticities of substitution σ_{kl}, $k,l \in \mathrm{N}_v$, then the error terms Δ_v, $v = 1,\ldots,r$, can be computed from the observed price changes. In order to determine the error term Δ_{ij} [see equation (20)] however, the additional condition $d_iC = -d_jC$ and $d_kC = 0$, $k \neq i,j$, had to be imposed. An arbitrary price change will in general not satisfy this condition, but is rather likely to be a change in most, if not all, prices. All we can do is to reduce by suitable normalization the price change so that it is limited to the tangent plane to the factor price frontier, i.e., so that total cost remains unchanged.

A more general approach which will allow for an arbitrary price change is needed. This will be developed in Section 9, but first we need to define a shadow elasticity of substitution for an arbitrary price change, and such a directional SES will be introduced in the next section.

8. A Directional Shadow Elasticity of Substitution

There are several possible definitions of the Shadow Elasticity of Substitution (SES). The usual one is formulated in terms of the percentage change in the input ratio resulting from a percentage change in the price ratio, holding everything else constant. Let

$$C = C(y;p_1,\ldots,p_m) \tag{27}$$

be a cost function defined as a function of the output rate y and the input prices p_i, $i = 1,\ldots,m$. Then the SES between the ith and the kth input is defined by

$$\mathrm{SES}_{ik} = -\frac{\mathrm{d}(C_i/C_k)}{C_i/C_k}\bigg/\frac{\mathrm{d}(p_i/p_k)}{p_i/p_k}, \qquad C_i = \frac{\partial C}{\partial p_i}, \tag{28}$$

holding total cost, y, and p_j, $j \neq i,k$, constant. Holding total cost and output constant insures that the resulting change represents a movement along the factor price frontier.

This definition, however, does not lend itself well to generalization when more than two prices change and we are interested in the curvature of the factor price frontier in the direction of this price change. The concept of a percentage change in an input ratio or a price ratio ceases to be very meaningful. But there is an alternative definition of the SES which can be more readily generalized. The factor price frontier defined by (27) holding total cost and output constant defines implicitly a function between any one price, say p_k, and the remaining prices;

$$p_k = R^k(p_1,\ldots,p_i,\ldots,p_m), \qquad i \neq k. \tag{29}$$

Holding p_j, $j \neq i,k$, constant, the function R^k defines a relationship between p_i and p_k, which will be a curve on the factor price frontier in the (i,k) plane. The shadow elasticity of substitution can now be defined by

$$\mathrm{SES}_{ik} = \frac{\partial^2 R^k/\partial p_i^2}{\partial R^k/\partial p_i}\bigg/\frac{(\partial/\partial p_i)(p_k/p_i)}{p_k/p_i}. \tag{30}$$

8.1. *Definition of DSES*

Let us assume that all prices change simultaneously in such a manner as to leave us on the same factor price frontier, i.e., output and total cost remain constant. We can define this price change by the following vector valued function of the scalar variable t (which may be identified as time),

$$\mathbf{p}(t) = \mathbf{p}^0 + t\mathbf{v}, \tag{31}$$

where $\mathbf{p}^0$ is the base point at which we will evaluate the directional SES. Around $t = 0$, (31) will describe a curve on the factor price frontier in the direction

$$\partial \mathbf{p}/\partial t = \mathbf{v},$$

where only $m-1$ of the components of $\mathbf{v}$ are free, the remaining, say v_k is determined by the condition that we remain on the factor price frontier, i.e.,

$$v_k = -\frac{1}{x_k}\sum_{\substack{i=1\\ i\neq k}}^{m} x_i v_i.$$

Let $R^k(p_1,\ldots,p_i,\ldots,p_m)$, $i\neq k$, be the function defined by (29), but now regarded as a function of t, then as t changes, (31) will determine a curve on the factor price frontier,

$$\mathbf{r}(t) = [p_1^0 + tv_1;\ldots;p_i^0 + tv_i;\ldots;R^k(p_1^0 + tv_1;\ldots; p_i^0 + tv_i;\ldots;p_m^0 + tv_m);\ldots;p_m^0 + tv_m], \qquad i\neq k, \tag{32}$$

and we are interested in the behavior of $\mathbf{r}(t)$ at the point $\mathbf{p}^0$ in the direction of the vector $\mathbf{v}$.

Let $\mathbf{x}^0$ be the vector of factor demand equations at $\mathbf{p}^0$, let

$$\alpha_i = x_i^0 v_i \Big/ \sum_{\substack{j=1\\ j\neq k}}^{m} x_j^0 v_j \qquad i = 1,\ldots,m, \quad i\neq k,$$

(note that the α_i's need not lie between 0 and 1), and define for each point $\mathbf{p}^0$ and each direction $\mathbf{v}$ the linearly homogeneous (since $\sum \alpha_i = 1$) "Cobb–Douglas" type price aggregate,

$$\phi^k(t) = \exp\left[\int \left(\sum_{\substack{i=1\\ i\neq k}}^{m} x_i^0 v_i \frac{v_i}{p_i} \Big/ \sum_{\substack{i=1\\ i\neq k}}^{m} x_i^0 v_i\right) \mathrm{d}t\right]$$

$$= \prod_{\substack{i=1\\ i\neq k}}^{m} [p_i(t)]^{\alpha_i}, \qquad v_k \neq 0. \tag{33}$$

Definition. Given the cost function (27) and the price function (29), the Directional Shadow Elasticity of Substitution in the direction $\mathbf{v}$ is

$$\mathrm{DSES}(\mathbf{v}) = \frac{\mathrm{d}^2 R^k/\mathrm{d}t^2}{\mathrm{d}R^k/\mathrm{d}t} \Big/ \frac{(\mathrm{d}/\mathrm{d}t)(p_k/\phi^k)}{p_k/\phi^k}, \tag{34}$$

holding output constant.

Theorem. Expressed in terms of the shadow elasticities of substitution σ_{ij}, the Directional SES can be written as

$$\mathrm{DSES}(\mathbf{v}) = -\frac{1}{2}\left(\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}(x_i p_i + x_j p_j)\sigma_{ij}\frac{v_i v_j}{p_i p_j}\bigg/\sum_{i=1}^{m} x_i v_i \frac{v_i}{p_i}\right), \tag{35}$$

subject to $\sum_{i=1}^{m} x_i v_i = 0$.

Proof: See Section 10.2.

Even though the definition of DSES(**v**) [equation (34)] was expressed in terms of a dependent price variable (in that case p_k), equation (35), which could also have been taken as the definition, is symmetric in its treatment of all the variables. This shows that the resulting value of the DSES(v) is independent of which price was chosen as the dependent variable in the definition.

8.2. *Properties of the DSES*

Lemma 1. The Directional SES is non-negative, i.e., DSES(**v**) > 0 for every vector **v**.

Proof: Since the cost function is concave in prices, the function R^k will be convex, and the quadratic form [(A.4) in Section 10.2] will be positive semidefinite, i.e.,

$$\frac{\mathrm{d}^2 R^k}{\mathrm{d}t^2} = \sum_{\substack{i=1\\ i\neq k}}^{m}\sum_{\substack{j=1\\ j\neq k}}^{m} R^k_{ij} v_i v_j \geq 0.$$

Using equations (A.3) and (A.6) from the proof of the theorem, the DSES(**v**) becomes

$$\begin{aligned}\mathrm{DSES}(\mathbf{v}) &= \left(\frac{\mathrm{d}^2 R^k}{\mathrm{d}t^2}\bigg/-\frac{1}{x_k}\sum_{\substack{i=1\\ i\neq k}}^{m} x_i v_i\right)\bigg/-\left(\sum_{i=1}^{m} x_i v_i \frac{v_i}{p_i}\bigg/\sum_{\substack{i=1\\ i\neq k}}^{m} x_i v_i\right)\\ &= \frac{\mathrm{d}^2 R^k}{\mathrm{d}t^2}\bigg/\frac{1}{x_k}\sum_{i=1}^{m}\frac{x_i}{p_i}(v_i)^2,\end{aligned}$$

where both numerator and denominator are positive. It may be noted that we have at the same time proved that the quadratic form of the substitution terms is negative semidefinite for every price change in the tangent plane, i.e.,

$$\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq k}}^{m} d_{ij}x_iv_ix_jv_j \leq 0 \quad \text{for every} \quad x_iv_i,$$

such that $\sum_{i=1}^{m} x_iv_i = 0$.

Define the coefficients

$$a_{ij}(\mathbf{v}) = -\frac{1}{2}\left((x_ip_i + x_jp_j)\frac{v_iv_j}{p_ip_j}\Big/\sum_{s=1}^{m} x_sv_s\frac{v_s}{p_s}\right),$$

$$i,j = 1,\ldots,m, \quad i \neq j. \tag{36}$$

We then have

$$\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij}(\mathbf{v})$$

$$= -\frac{1}{2}\left[\left(\sum_{j=1}^{m}\frac{v_j}{p_j}\sum_{\substack{i=1\\ i\neq j}}^{m} x_iv_i + \sum_{i=1}^{m}\frac{v_i}{p_i}\sum_{\substack{j=1\\ j\neq i}}^{m} x_jv_j\right)\Big/\sum_{s=1}^{m} x_sv_s\frac{v_s}{p_s}\right] = 1,$$

and the directional SES can be written as

$$\text{DSES}(\mathbf{v}) = \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij}(\mathbf{v})\sigma_{ij}, \tag{37}$$

i.e., the DSES is a weighted average (though not necessarily with positive weights) of the individual shadow elasticities of substitution. An SES will have a negative weight if the related prices change in the same direction, and a positive weight if they change in opposite direction. If only two prices, say p_i and p_j, were to change, then the DSES reduces to the SES, i.e., we have the following lemma:

Lemma 2. Let $\mathbf{v}' = [0,\ldots,v_i,\ldots,0,v_j,\ldots,0]$, i.e., $\mathbf{v}'$ is a direction vector with only its ith and jth components different from zero, then

$$\text{DSES}(\mathbf{v}') = \sigma_{ij}.$$

Proof: We must have that $v_j = -(x_i/x_j)v_i$ to insure that we stay on the factor price frontier; and with $v_k = 0$, $k \neq i,j$, (35) becomes

$$\text{DSES}(\mathbf{v}') = -\frac{(x_ip_i + x_jp_j)\sigma_{ij}(v_i/p_i)(-x_iv_i/x_jp_j)}{x_iv_i(v_i/p_i) + (x_jx_i^2/p_jx_j^2)v_i^2} = \sigma_{ij}.$$

We further have:

> *Lemma* 3. At any point in the price space, the DSES(v) is the same in every direction if and only if the SES's are all equal, and DSES(v) and SES will then have the same value.[26]

Proof: If $\sigma_{ij} = \sigma$ for every $i,j = 1,\ldots,\ i \neq j$, then

$$\mathrm{DSES}(\mathbf{v}) = \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} a_{ij}(\mathbf{v})\sigma_{ij} = \sigma \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} a_{ij}(\mathbf{v}) = \sigma.$$

The "only if" part follows from Lemma 2.

9. Inconsistent Directional Shadow Elasticity of Substitution

We will now return to the task left unfinished at the end of Section 7. Proceeding in a manner analogous to Section 8, we can use the factor price frontier of the inconsistently aggregated cost function B [equation (16)] to define a function R^k between the inconsistent price aggregate π^k and the remaining inconsistent price aggregates π^i, $i \neq k$, i.e.,

$$\pi^k = R^k(\pi^1,\ldots,\pi^i,\ldots,\pi^r). \tag{38}$$

Further, assuming $\mathrm{d}\pi^k \neq 0$, define the price aggregate

$$\phi^k = \prod_{\substack{i=1 \\ i \neq k}}^{r} (\pi^i)^{\alpha_i},$$

where

$$\alpha_i = B_i\, \mathrm{d}\pi^i \Big/ \sum_{\substack{j=1 \\ j \neq k}}^{r} B_j\, \mathrm{d}\pi^j = \mathrm{d}_i C \Big/ \sum_{\substack{j=1 \\ j \neq k}}^{r} \mathrm{d}_j C. \tag{39}$$

Then the inconsistently aggregated directional shadow elasticity of substitution can be defined [analogously to equation (34)] as

$$\mathrm{DSES}^B(\mathrm{d}\boldsymbol{\pi}) = \frac{\mathrm{d}^2 R^k}{\mathrm{d}R^k} \Big/ \frac{\mathrm{d}(p_k/\phi^k)}{p_k/\phi^k}. \tag{40}$$

[26] It was the *a priori* requirement of the "if" part of this lemma, or equivalently the condition that

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} a_{ij}(v) = 1,$$

that led to the definition of the price aggregate ϕ^k along the lines of equation (33).

Lemma 4. Expressed in terms of the shadow elasticities of substitution Σ_{ij} of the consistently aggregated cost function $\mathscr{C}(y;\rho^1,\ldots,\rho^r)$ and the error terms Δ_v [see equation (21)], the inconsistently aggregated DSES can be written as

$$\mathrm{DSES}^B(\mathrm{d}\boldsymbol{\pi}) = -\frac{1}{2}\Big(\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}(C^i + C^j)\Sigma_{ij}\frac{\mathrm{d}_iC}{C^i}\frac{\mathrm{d}_jC}{C^j}\Big)\Big/\Big(\sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}\Big) + \Big(\sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}\Delta_i\Big)\Big/\Big(\sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}\Big). \tag{41}$$

Proof: See proof of theorem above and Frenger (1975, pp. 110–119).

The first term on the r.h.s. of equation (41) is the $\mathrm{DSES}^{\mathscr{C}}(\mathrm{d}\boldsymbol{\pi})$, the consistently aggregated DSES of the cost function $\mathscr{C}$. Define the directional error term

$$\Delta(\mathrm{d}\boldsymbol{\pi}) = \sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}\Delta_i\Big/\sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}. \tag{42}$$

The equation (41) can be written more concisely as

$$\mathrm{DSES}^B(\mathrm{d}\boldsymbol{\pi}) = \mathrm{DSES}^{\mathscr{C}}(\mathrm{d}\boldsymbol{\pi}) + \Delta(\mathrm{d}\boldsymbol{\pi}). \tag{43}$$

Assume that only the ith and jth component of $\mathrm{d}\boldsymbol{\pi}$ change, (i.e., $B_i\,\mathrm{d}\pi^i = -B_j\,\mathrm{d}\pi^j$, and $\mathrm{d}\pi^k = 0$, $k\neq i,j$), and call this vector $\mathrm{d}\boldsymbol{\pi}'$, then

$$\Delta(\mathrm{d}\boldsymbol{\pi}') = \frac{\Delta_i/C^i + \Delta_j/C^j}{1/C^i + 1/C^j} = \Delta_{ij}. \tag{44}$$

Combining equation (44) and Lemma 2, we have just proved the following lemma:

Lemma 5. Let $\mathrm{d}\boldsymbol{\pi}'$ be defined as above, then the inconsistently aggregated DSES reduces to the inconsistently aggregated SES, i.e.,

$$\mathrm{DSES}^B(\mathrm{d}\boldsymbol{\pi}') = \mathrm{SES}^B_{ij}, \tag{45}$$

and the expression for SES^B_{ij} is given by equation (20). The converse is also true, i.e.:

Lemma 6. Let $\mathrm{d}\boldsymbol{\pi}$ be any price change, then $\mathrm{DSES}^B(\mathrm{d}\boldsymbol{\pi})$ can be obtained directly from SES^B_{ij}, $i,j = 1,\ldots,r$, by applying equation (35), i.e.,

$$\mathrm{DSES}^B(\mathrm{d}\pi) = -\frac{1}{2}\left(\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}(C^i + C^j)\mathrm{SES}^B_{ij}\frac{\mathrm{d}_iC}{C^i}\frac{\mathrm{d}_jC}{C^j}\Big/\sum_{i=1}^{r}\mathrm{d}_iC\frac{\mathrm{d}_iC}{C^i}\right). \tag{46}$$

Proof: Just substitute (20) into the r.h.s. of (46), and (41) will result.

The estimates presented in Section 4 of this paper all include these aggregation errors, i.e., all the estimates refer to the inconsistently aggregated function B and not to the true aggregated cost function $\mathscr{C}$. (We will assume that the disaggregated cost function is weakly separable so that consistent aggregates do exist. This assumption can best be defended as a first approximation in order to make the following application possible.)

As explained in Section 7 [equation (19)] it is reasonable to expect the Paasche–Laspeyres aggregates to satisfy the first-order condition for consistent price aggregates [equation (18)] for the base period of the price index, in our case 1955. It is for this reason that the following computations are limited to the price change between 1955 and 1956.

The SES presented in Table 1 are the inconsistently aggregated SES, i.e., SES^B_{ij}, and using equation (46) we can then compute the inconsistently aggregated DSES in the direction of the 1955–1956 price change. These are[27]

Textiles:

$$\mathrm{DSES}^B(\mathrm{d}\pi) = 0.1347,$$

Construction:

$$\mathrm{DSES}^B(\mathrm{d}\pi) = 0.8088. \tag{47}$$

By assuming (for the sake of illustration) that all the elasticities of substitution within an aggregation group are equal, i.e., $\sigma_{ij} = \sigma_v$ for every $i,j \in N_v$, we can use (24) to derive the sectoral error terms and then (42) to compute the directional error term (in this last step I am assuming that $\sigma_v = \sigma$ for $v = 1,\ldots,r$). We then have

[27]In the case of metals, the estimate was performed on the principal disaggregated sectors, so no aggregation was involved and $\Delta(\mathrm{d}\pi) = 0$. The sector has therefore been excluded from the following computations.

Textiles:

$$\Delta_D = -1.2350\sigma_D, \qquad \Delta_F = -0.0841\sigma_F,$$
$$\Delta(\mathrm{d}\boldsymbol{\pi}) = -0.0566\sigma,$$

Construction:

$$\Delta_1 = -0.1510\sigma_1, \qquad \Delta_2 = -2.8375\sigma_2, \qquad \Delta_3 = -1.1812\sigma_3,$$
$$(\Delta_5 = 0 \quad \text{since} \quad \mathrm{d}_5 C = 0),$$
$$\Delta(\mathrm{d}\boldsymbol{\pi}) = -0.2655\sigma.$$

Note also that we did not conduct any aggregation within the sectors capital and labor, and hence $\Delta_k = \Delta_L = 0$ for both textiles and construction.

Rewriting equation (43) as

$$\mathrm{DSES}^{\mathscr{C}}(\mathrm{d}\boldsymbol{\pi}) = \mathrm{DSES}^{B}(\mathrm{d}\boldsymbol{\pi}) - \Delta(\mathrm{d}\boldsymbol{\pi}),$$

we have the expression for the consistent directional SES, and it is

Textiles:

$$\mathrm{DSES}^{\mathscr{C}}(\mathrm{d}\boldsymbol{\pi}) = 0.1347 + 0.0566\sigma,$$

Construction:

$$\mathrm{DSES}^{\mathscr{C}}(\mathrm{d}\boldsymbol{\pi}) = 0.8088 + 0.2655\sigma. \tag{48}$$

These figures indicate that if $\sigma = 1$ the inconsistent aggregation would result in an underestimate of the "true" directional SES of 30% for textiles and 25% for construction, in the direction of the 1955–1956 price change, thus confirming the suspicion expressed at the beginning of Section 7.

These last computations are only intended as a realistic example of how inconsistent aggregation might affect the value of the substitution parameters of the cost function. It should also be remembered that the analysis of the last three sections has been limited to small changes around the base point, while the estimates of Table 1 are obviously affected by the behavior of the disaggregated variables over the entire sample period.

These arguments do, however, point out some of the difficulties involved in reaching good estimates of the aggregate elasticities of substitution, they show how serious biases can arise, and the need for giving special attention to the aggregation elasticities of substitution. And, lastly, they point out that it is not sufficient for the production structure to be separable (and weak separability of the aggregated input sets was assumed above), but we must also know the disaggregated elasticities of substitution in order to estimate correctly the aggregate elasticities of substitution.

10. Appendix

10.1. *The Industrial Classification and Tables*

This section explains the classification system used in the Norwegian input–output tables [Norway (1965)] and the abbreviations used in the text and the following tables.

The inputs are divided according to whether their origin is domestic or foreign, and the following numerical code is used:

Industry code		
Domestic	Foreign	Industry
11	61	Agriculture and dairy products
12	62	Food industries, excl. dairy products, canning, and fish processing
13	63	Textiles, footwear, and other apparel
14	64	Other consumer goods
21	71	Manufacturing of investment goods
22		Construction
31	81	Forestry, wood pulp, paper, and paper products
32	83	Fish and fish products, whaling
33	83	Metal mining, metals, ferro-alloys fertilizers, carbide
41	91	Water transport
53	93	Electricity supply
54	94	Real estate, dwellings
55	95	Trade and transport, excl. water transport
56	96	Service industries n.e.c.

It has been necessary to aggregate several input sectors prior to estimation. The most common procedure has been to aggregate the inputs according to the one digit classification system, and over domestic and imported commodities, giving rise to the following industrial sectors:

(1) Consumer goods
(2) Capital goods
(3) Export goods
(4) Water transport
(5) Service industries

No direct deliveries from Sector 4 appear in the three industries presently studied. The inputs from Sector 08, unspecified deliveries, have for simplicity always been assumed to be used in fixed proportions to output, and thus ignored in all the estimates. In the text and in the tables the industry code numer is preceded by an x to indicate an input from the xth sector or by a p to indicate it is the price index of the input from the xth sector.

The following special aggregates were used in the textile industry:

(i) All domestically produced inputs aggregated into xD and all imported inputs aggregated to xF.
(ii) All domestic sectors, except Sector 14, aggregated into xrD, while all imported sectors, except Sectors 61, 63, and 64 were aggregated into xrF.

In metals the principal inputs were included in the G.L. formulation, while the inputs from the other sectors were assumed used in fixed proportions to output.

The estimates are for the full period 1949–1961; and were estimated using generalized least squares (GLSQ) [see equation (9) in Section 4]. The matrices below present the estimated b_{ij} coefficients with their standard deviation in parenthesis below the estimate. Each row thus represents the factor demand equation for the input, whose symbol is given on the left. The column heading represents the numerator p_j in the relative price term $(p_j/p_i)^{1/2}$, the denumerator being the same for each row. Only the upper diagonal elements are shown in those cases where the symmetry constraint was imposed.

The tables give the square of the correlation coefficient (R^2), the Durbin–Watson statistic (DW), and the sum of squared residuals (SSR)

TABLE A.1

Textiles – Symmetric *B* matrix; 1949–1961.

	pD	*pF*	*pKB*	*pL*
xD	−0.2000 (0.2498)	0.1013 (0.1699)	0.0603 (0.0466)	0.1671 (0.0993)
xF		0.2318 (0.1247)	−0.1124 (0.0405)	0.1499 (0.0704)
xKB			0.0407 (0.0510)	0.1387 (0.0390)
xL				−0.1104 (0.0623)

	$\sum_{j \neq i} b_{ij}$	$\sum_{j=1}^{n} b_{ij}$	SSR
xD	0.3287	0.1287	179.8×10^{-5}
xF	0.1389	0.3707	487.8
xKB	0.0866	0.1273	136.2
xL	0.4557	0.3453	692.9
			1496.7

$R^2 = 0.9799$ DW = 1.5913 SSR = 1496.7×10^{-5}

TABLE A.2

Textiles – Symmetric *B* matrix, 6 intermediate inputs; 1949–1961.

	p14	*prD*	*p61*	*p63*	*p64*	*prF*
x14	−0.0064 (0.0871)	−0.0622 (0.0486)	−0.1073 (0.0588)	0.1799 (0.0312)	−0.0022 (0.0745)	0.0581 (0.0268)
xrD		0.0512 (0.0568)	0.0209 (0.0394)	−0.0386 (0.0289)	0.0787 (0.0557)	0.0179 (0.0285)
x61			−0.0153 (0.0519)	0.2327 (0.0292)	−0.0260 (0.0522)	−0.0490 (0.0236)
x63				−0.1669 (0.0330)	0.0334 (0.0318)	0.0042 (0.0209)
x64					0.0274 (0.0894)	−0.0491 (0.0287)
xrF						0.0239 (0.0279)

	$\sum_{j \neq i} b_{ij}$	$\sum_{j=1}^{n} b_{ij}$	SSR
x14	0.0664	0.0600	38.0×10^{-5}
xrd	0.0167	0.0679	34.1
x61	0.0714	0.0561	40.2
x63	0.4117	0.2448	153.2
x64	0.0348	0.0622	13.8
xrF	−0.0179	0.0060	6.1
			285.4

$R^2 = 0.9924$ DW = 2.3252 SSR = 285.4×10^{-5}

for each of the estimated equations. The sum of squared residuals is shown for each factor demand equation also when the B matrix was constrained to be symmetric, even though the factor demand equations were estimated simultaneously.

Further, for those cases where the matrix B is constrained to be symmetric, the tables present two coefficient sums:

$\sum_{j \neq 1} b_{ij}$ – The sum of the off-diagonal coefficients. For the base year 1955 (with $p_i = 1.00$) this becomes a test of the concavity of the estimated cost function [see equations (4) and (12)].

$\sum_{j=1}^{n} b_{ij}$ – This is just the fitted value of the ith factor demand equation for the base year. It shows the magnitude of the input–output coefficient of that sector, and is a test of the theoretical requirement that the cost function be non-decreasing in the input prices [see equation (11)].

TABLE A.3
Construction – Unconstrained B matrix; 1949–1961.

	$p1$	$p2$	$p3$	$p5$	pK	pL
$x1$	−0.1963	0.0567	0.0506	0.1976	0.0174	0.0357
	(0.1084)	(0.1487)	(0.1269)	(0.1929)	(0.0379)	(0.2741)
$x2$	0.1896	−0.3336	0.4497	0.1952	−0.0398	−0.2411
	(0.1803)	(0.2467)	(0.2117)	(0.3251)	(0.0634)	(0.4620)
$x3$	0.0232	0.1408	0.3083	0.1933	−0.0874	−0.4012
	(0.1038)	(0.1443)	(0.1214)	(0.1883)	(0.0367)	(0.2679)
$x5$	−0.0221	0.0817	−0.0995	0.0070	0.0149	0.0524
	(0.0250)	(0.0354)	(0.0300)	(0.0470)	(0.0093)	(0.0670)
xK	−0.1289	0.0562	−0.0801	0.1032	0.0104	0.0605
	(0.0388)	(0.0499)	(0.0410)	(0.0619)	(0.0118)	(0.0862)
xL	0.3964	0.1336	−0.3926	−0.4296	0.0223	0.6043
	(0.0787)	(0.1119)	(0.0946)	(0.1484)	(0.0292)	(0.2117)

	R^2	DW	SSR
$x1$	0.9474	2.0369	27.1×10^{-5}
$x2$	0.6717	2.4045	78.0
$x3$	0.4651	2.1236	25.8
$x5$	0.7976	2.1831	1.6
xK	0.9880	2.0927	2.7
xL	0.9830	2.9608	16.1
			151.3

TABLE A.4

Construction – Symmetric *B* matrix; 1949–1961.

	p1	*p2*	*p3*	*p5*	*pK*	*pL*
x1	−0.0520 (0.0931)	−0.0991 (0.0739)	−0.0707 (0.0821)	0.0440 (0.0236)	0.0149 (0.0286)	0.3250 (0.0500)
x2		0.2942 (0.1065)	0.2381 (0.0874)	−0.0064 (0.0285)	−0.0086 (0.0259)	−0.1911 (0.0588)
x3			0.1515 (0.1038)	−0.0084 (0.0252)	−0.0375 (0.0275)	−0.0931 (0.0611)
x5				0.0421 (0.0243)	−0.0044 (0.0113)	−0.0332 (0.0343)
xK					−0.0314 (0.0137)	0.0903 (0.0183)
xL						0.2277 (0.0665)

	$\sum_{j \neq i} b_{ij}$	$\sum_{j=1}^{n} b_{ij}$	SSR
x1	0.2141	0.1621	77.4×10^{-5}
x2	−0.0671	0.2271	466.9
x3	0.0284	0.1799	48.2
x5	−0.0084	0.0337	6.0
xK	0.0547	0.0233	15.0
xL	0.0979	0.3256	176.6
			790.1

TABLE A.5

Metals – Unconstrained *B* matrix, with "user cost" of capital; 1949–1961.

	p14	*p53*	*p64*	*p83*	*pKA*	*pL*
x14	−0.0484 (0.0358)	−0.0639 (0.0686)	−0.0616 (0.0629)	0.0168 (0.0380)	0.1144 (0.0556)	0.0693 (0.0364)
x53	−0.0244 (0.0367)	−0.1103 (0.0717)	−0.0913 (0.0636)	0.0504 (0.0845)	0.0845 (0.0562)	0.1431 (0.0377)
x64	−0.2652 (0.0969)	0.2363 (0.1816)	−0.1409 (0.1716)	0.1989 (0.1037)	0.1824 (0.1482)	−0.1347 (0.0975)
x83	0.1557 (0.2516)	−0.5851 (0.4644)	0.2594 (0.4428)	−0.1022 (0.2680)	0.0545 (0.3849)	0.4983 (0.2486)
xKA	0.0424 (0.0584)	−0.0381 (0.1119)	−0.0267 (0.1018)	0.0687 (0.0615)	0.0023 (0.0893)	0.0951 (0.0595)
xL	0.0336 (0.0895)	0.1531 (0.1784)	0.2242 (0.1542)	0.1031 (0.0929)	−0.1272 (0.1381)	−0.1944 (0.0930)

	R^2	DW	SSR
x14	0.8930	1.5043	9.1×10^{-5}
x53	0.8363	3.2460	9.8
x64	0.8488	2.4860	68.7
x83	0.6217	1.8320	443.1
xKA	0.5344	2.1299	24.0
xL	0.9651	1.8873	60.4
			615.1

TABLE A.6
Metals – Symmetric *B* matrix, with "user cost" of capital; 1949–1961.

	p14	*p53*	*p64*	*p83*	*pKA*	*pL*
x14	−0.0056 (0.0331)	−0.0399 (0.0316)	−0.1863 (0.0488)	0.1087 (0.0327)	0.0509 (0.0287)	0.0972 (0.0381)
x53		−0.0623 (0.0643)	0.0691 (0.0541)	−0.0534 (0.0392)	0.0469 (0.0459)	0.0945 (0.0440)
x64			−0.1902 (0.1049)	0.2487 (0.0710)	0.1144 (0.0483)	0.0213 (0.0490)
x83				−0.1245 (0.0739)	−0.0588 (0.0379)	0.1589 (0.0434)
xKA					−0.0111 (0.0474)	0.0034 (0.0361)
xL						−0.1868 (0.0466)

	$\sum_{j \neq i} b_{ij}$	$\sum_{j=1}^{n} b_{ij}$	SSR
x14	0.0306	0.0250	18.4×10^{-5}
x53	0.1181	0.0558	28.6
x64	0.2672	0.0770	110.3
x83	0.4041	0.2796	573.7
xKA	0.1563	0.1457	53.6
xL	0.3753	0.1885	83.7
			868.3

TABLE A.7
Textiles – Predicted 1961 input–output coefficients.

		1961 relative prediction errors		
	1961 coeff.	1960 coeff.	1949–60 avg. coeff.	Avg. coeff. *w*/trend
xD[a]	0.1339[b]	0.0254	−0.0083	−0.0151
xF	0.3973	−0.1223	−0.0903	−0.0814
xK	0.1214	−0.0008	0.0189	0.1071
xL	0.2665	0.0473	0.3568	0.0131
Avg.		0.0490	0.1186	0.0542
Wgt. avg.		0.0704	0.1462	0.0553
x14	0.0654	−0.0535		
xrD	0.0685	0.1007		
x61	0.0573	−0.1204		
x63	0.2640	−0.1534		
x64	0.0648	0.0448		
xrF	0.0112	−0.3750		
Avg.		0.1413		
Wght. avg.		0.1222		

TABLE A.7 (continued)

		1960–61 coeff. change
	Symm. GL	Symm. GL
xD	0.2636	0.1583
xF	−0.0489	−0.1162
xKB	0.1450	0.0008
xL	0.0841	−0.0499
Avg.	0.1354	0.0813
Wght. avg.	0.1031	0.0879
x14	−0.1131	−0.0336
xrD	0.0832	0.1080
x61	−0.0733	−0.1606
x63	−0.1269	−0.1530
x64	0.0478	0.0324
xrF	−0.1964	−0.2857
Avg.	0.1068	0.1288
Wght. avg.	0.1056	0.1214

[a]See the industry classification for explanation of code.
[b]These are the actual 1961 input–output coefficients.

TABLE A.8
Construction – Predicted 1961 input–output coefficients.

		1961 relative prediction errors		
	1961 coeff.	1960 coeff.	1949–60 avg. coeff.	Avg. coeff. w/trend
x1[a]	0.1963[b]	−0.0680	−0.1972	−0.0158
x2	0.2478	−0.0588	−0.1217	−0.0483
x3	0.1672	0.0794	0.0259	0.0941
x5	0.0318	0.0540	0.0414	0.0178
xK	0.0486	−0.1122	−0.5252	−0.1337
xL	0.2925	0.0295	0.1040	0.0151
Avg.		0.0670	0.1629	0.0541
Wght. avg.		0.0579	0.1325	0.0429

	Unsymm. GL	Symm. GL
x1	0.0010	−0.0051
x2	−0.0606	−0.0052
x3	0.0385	0.0353
x5	0.0207	−0.0031
xK	−0.0575	−0.2140
xL	−0.0071	0.0232
Avg.	0.0309	0.0472
Wght. avg.	0.0276	0.0259

TABLE A.8 (continued)

	1960–61 coeff. change	
	Unsymm. GL	Symm. GL
x1	−0.0056	−0.0020
x2	−0.0420	−0.0032
x3	0.0455	0.0490
x5	0.0283	0.0283
xK	−0.0288	−0.1091
xL	−0.0198	−0.0017
Avg.	0.0309	0.0322
Wght. avg.	0.0276	0.0163

[a]See the industry classification for explanation of code.
[b]These are the actual 1961 input–output coefficients.

TABLE A.9
Metals – Predicted 1961 input–output coefficients.

		1961 relative prediction errors		
	1961 coeff.	1960 coeff.	1949–60 avg. coeff.	Avg. coeff. *w*/trend
x14[a]	0.0492[b]	−0.2297	−0.4878	−0.3394
x53	0.0633	0.0284	−0.1817	−0.0348
x64	0.1429	−0.5416	−0.4612	−0.4843
x83	0.2405	0.3231	0.1343	0.3489
xKA[c]	0.1339	0.0545	0.0739	0.1060
xKB[c]	0.2636	0.0545	0.0739	0.1060
xL	0.1410	0.0894	0.4553	0.0298
Avg.		0.2100	0.2990	0.2239
Wght. avg. *w*/*KA*		0.2440	0.2696	0.2470
Wght. avg. *w*/*KB*		0.2164	0.2414	0.2267
Wght. avg. intermed. inputs only		0.3385	0.2696	0.3469

	Symmetric Generalized Leontief		
	"User cost"	"Rate of return"	Intermed. inputs only
x14	−0.4065	−0.3455	−0.8415
x53	−0.1864	0.2370	0.0126
x64	0.5444	−0.4920	−0.6718
x83	0.3326	0.2960	0.2603
xKA	0.0321		
xKB		−0.1885	
xL	0.1631	0.1610	
Avg.	0.2775	0.2867	0.4466
Wght. avg. *w*/*KA*	0.2813		
Wght. avg. *w*/*KB*		0.2764	
Wght. avg. intermed. inputs only	0.3823	0.3499	0.4049

TABLE A.9 (continued)

	1960–61 coeff. change Symmetric Generalized Leontief		
	"User cost"	"Rate of return"	Interm. inputs only
x14	−0.2825	−0.1748	−0.5122
x53	−0.1501	−0.1706	0.2859
x64	−0.5892	−0.5374	−0.6704
x83	0.3601	0.3289	0.3721
xKA	0.0314		
xKB		−0.2246	
xL	0.1440	0.1532	
Avg.	0.2596	0.2649	0.4602
Wght. avg. *w/KA*	0.2837		
Wght. avg. *w/KB*		0.2844	
Wght. avg. intermed. inputs only	0.3916	0.3535	0.4610

[a]See the industry classification for explanation of code.
[b]These are the actual 1961 input–output coefficients.
[c]*xKA* represents an estimate using the "user cost" of capital, *xKB* using the "rate of return" cost of capital.

10.2. *Proof of Theorem*

Rewrite the cost function (27) as

$$C = C(y;p_1,\ldots,R^k,\ldots,p_i,\ldots,p_m), \qquad i \neq k.$$

Differentiating implicitly w.r.t. p_i gives

$$0 = C_i + C_k \frac{\partial R^k}{\partial p_i} = x_i + x_k R_i^k,$$

or

$$R_i^k = -\frac{x_i}{x_k} = -\frac{C_i(y;p_1,\ldots,R^k,\ldots,p_m)}{C_k(y;p_1,\ldots,R^k,\ldots,p_m)}. \tag{A.1}$$

Differentiating R_i^k w.r.t. p_j, $j \neq k$, gives

$$R_{ij}^k = -\frac{1}{C_k^2}[C_k(C_{ij} + C_{ik}R_j^k) - C_i(C_{kj} + C_{kk}R_j^k)],$$

and substituting for C_k, C_i, and R_j^k,

$$R_{ij}^k = \frac{x_i x_j}{x_k}\left[\frac{C_{ik}}{x_i x_k} + \frac{C_{jk}}{x_j x_k} - \frac{C_{ij}}{x_i x_j} - \frac{C_{kk}}{x_k x_k}\right]. \tag{A.2}$$

Expressed in terms of the substitution terms d_{ij}, where

$$d_{ij} = -\frac{C_{ii}}{x_i^2} + 2\frac{C_{ij}}{x_i x_j} - \frac{C_{jj}}{x_j^2}, \qquad i \neq j,$$

the second derivatives become

$$R_{ij}^k = \frac{1}{2}\frac{x_i x_j}{x_k}[d_{ik} + d_{jk} - d_{ij}], \quad i \neq j,$$

$$R_{ii}^k = \frac{x_i^2}{x_k} d_{ik}.$$

Returning to the derivatives of R^k w.r.t. t we have

$$\frac{\mathrm{d}R^k}{\mathrm{d}t} = \sum_{\substack{i=1\\i\neq k}}^{m} R_i^k \frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_{\substack{i=1\\i\neq k}}^{m} R_i^k v_i = -\frac{1}{x_k}\sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i, \tag{A.3}$$

$$\begin{aligned}
\frac{\mathrm{d}^2R^k}{\mathrm{d}t^2} &= \sum_{\substack{i=1\\i\neq k}}^{m}\sum_{\substack{j=1\\j\neq k}}^{m} R_{ij}^k v_i v_j \\
&= \frac{1}{2x_k}\sum_{\substack{i=1\\i\neq k}}^{m}\sum_{\substack{j=1\\j\neq k,i}}^{m} x_i x_j (d_{ik} + d_{jk} - d_{ij}) v_i v_j + \frac{1}{x_k}\sum_{\substack{i=1\\i\neq k}}^{m} x_i^2 d_{ik} v_i^2 \\
&= -\frac{1}{2x_k}\left(\sum_{\substack{i=1\\i\neq k}}^{m}\sum_{\substack{j=1\\j\neq k,i}}^{m} d_{ij} x_i v_i x_j v_j - \sum_{\substack{i=1\\i\neq k}}^{m} d_{ik} x_i v_i \sum_{\substack{j=1\\j\neq k,i}}^{m} x_j v_j \right.\\
&\qquad \left. - \sum_{\substack{j=1\\j\neq i}}^{m} d_{jk} x_j v_j \sum_{\substack{i=1\\i\neq k,j}}^{m} x_i v_i - 2\sum_{\substack{i=1\\i\neq k}}^{m} d_{ik} x_i^2 v_i^2\right) \\
&= -\frac{1}{2x_k}\sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m} d_{ij} x_i v_i x_j v_j.
\end{aligned} \tag{A.4}$$

In terms of the shadow elasticities of substitution

$$\sigma_{ij} = \frac{x_i p_i x_j p_j}{x_i p_i + x_j p_j} d_{ij},$$

the second derivative of R^k becomes

$$\frac{\mathrm{d}^2R^k}{\mathrm{d}t^2} = -\frac{1}{2x_k}\sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m} (x_i p_i + x_j p_j)\sigma_{ij}\frac{v_i v_j}{p_i p_j},$$

and the numerator of DSES(v) can be written as

$$\frac{\mathrm{d}^2R^k}{\mathrm{d}t^2}\bigg/\frac{\mathrm{d}R^k}{\mathrm{d}t} = \frac{1}{2}\left(\sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m} (x_i p_i + x_j p_j)\sigma_{ij}\frac{v_i v_j}{p_i p_j}\bigg/\sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i\right). \tag{A.5}$$

Since

$$\frac{\mathrm{d}\phi^k}{\mathrm{d}t} = \sum_{\substack{i=1\\i\neq k}}^{m} \frac{\partial\phi^k}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} = \phi^k \sum_{\substack{i=1\\i\neq k}}^{m} \alpha_i \frac{v_i}{p_i},$$

the denumerator of DSES($\mathbf{v}$) becomes

$$\begin{aligned}\frac{\phi^k}{p_k}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_k}{\phi^k}\right) &= \frac{1}{p_k}\frac{\mathrm{d}p_k}{\mathrm{d}t} - \frac{1}{\phi^k}\frac{\mathrm{d}\phi^k}{\mathrm{d}t}\\ &= \left(\frac{v_k}{p_k}(-x_k v_k) - \sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i \frac{v_i}{p_i}\right)\Big/\sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i\\ &= -\sum_{i=1}^{m} x_i v_i \frac{v_i}{p_i}\Big/\sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i,\end{aligned} \tag{A.6}$$

where

$$v_k = -\frac{1}{x_k}\sum_{\substack{i=1\\i\neq k}}^{m} x_i v_i.$$

Taking the ratio of (A.5) over (A.6) gives (35).