Chapter III.2

HICKS' AGGREGATION THEOREM AND THE EXISTENCE OF A REAL VALUE-ADDED FUNCTION*

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1. Introduction

The received theory of consumer demand, which deals with the case of a decision-making unit which maximizes a utility function subject to a budget constraint, treats "commodities" as well-defined and distinct. However, when we look at empirical applications of this theory, we find that the "commodities" are actually *aggregates* of distinct goods and services, and thus we may ask whether the received theory of consumer demand has any empirical relevance, or alternatively, we must look for conditions which will justify using aggregate commodities in place of micro commodities.

It turns out that this use of aggregates in the theory of consumer demand can be justified, provided that all price changes within an aggregate are proportional. This result is known as *Hicks' Aggregation Theorem*. Somewhat imprecise statements and proofs of this theorem may be found in Hicks (1946, pp. 312–313) and Wold (1953, pp. 109–110), while a rather more precise statement of the theorem may be found in Gorman (1953, pp. 76–77).

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In Section 2 of this chapter, we will state and prove a version[1] of Hicks' Aggregation Theorem which is similar to Gorman's version, except that we relax his rather strong regularity conditions on the micro utility function.[2] We then ask the obvious question: what are the properties of the "aggregated" or "macro" utility function when prices within a subgroup do *not* vary in strict proportion? In Section 3, we answer this question and we note that there is a duality between direct "micro" utility functions and "macro" [or "conditional indirect" Pollak's (1969) terminology or "variable indirect" using Epstein's (1975) terminology] utility functions.

Differentiation of a conditional indirect utility function with respect to prices yields Pollak's (1969) conditional demand functions. We exhibit some convenient functional forms for the conditional indirect utility function and derive the corresponding conditional demand functions. If we econometrically estimate the parameters of these conditional demand functions and if all of the parameters of the conditional indirect utility function appear in these demand functions, then by duality, we will have estimated a representation of the consumer's direct utility function. Thus if we identify the conditional goods as public goods,[3] an appropriate choice of functional form for the conditional indirect utility function can be used to solve one of the most vexing problems in the public finance literature: how can we get people to reveal their preferences about public goods? This use of conditional demand functions to solve the "free rider" problem was suggested by Pollak (1969, p. 63).

We conclude Section 3 by establishing the implications of concavity of the direct utility function on the conditional indirect utility function, and the implications of convexity in prices of the conditional indirect utility function on the direct utility function. These implications are useful in the analysis of risk aversion in a many-commodity world [e.g., see Stiglitz (1969) and Hanoch (1977)].

In Section 4 of this chapter, we develop an analogue to Hicks' Aggregation Theorem in the context of production theory. More

[1]Our treatment of Hicks' Aggregation Theorem largely parallels that of Pollak (1969, pp. 74–75) and the Arrow–Hahn (1971, pp. 144–145) treatment of the first period derived utility function which arose in their treatment of the temporary equilibrium.

[2]Gorman's regularity conditions on the micro utility function are: f is a continuous, strictly increasing, strictly quasi-concave, once differentiable function defined over the non-negative orthant which also has the property that each indifference surface does not intersect the coordinate planes.

[3]Samuelson (1969a, Sec. 5) defines a public good as one that enters two or more persons' utility.

specifically, we reconsider the old problem[4] of attempting to represent the "real output" of an industry by using some sort of deflated value-added concept. This deflation problem is of some empirical importance, since virtually all (non-fixed coefficients) production function studies use deflated value-added (rather than deflated gross output) as their measure of real output. How has this substitution of value-added for gross output been justified in the context of production function studies?

Consider the following highly aggregated example. Let Y represent real gross output, K real capital input, L labour input, and M inputs of new materials, energy and other intermediate commodities. Suppose that the technology of the firm or industry can be represented by the production function f where $Y = f(K,L,M)$.

One method of justifying the substitution of value-added for gross output has been to assume that the production function f is weakly separable[5] and can thus be written as $f(K,L,M) \equiv g(h(K,L),M)$ in which case we can identify $h(K,L)$ as real value-added. This *separability* approach has been studied by Corden (1969), Sims (1969) and Arrow (1974), and we will not pursue it in the present paper.[6]

Another approach to the problem of measuring real value-added has been to define nominal value-added V as the solution to a profit maximization problem and then to apply Hicks' Aggregation Theorem. That is, we may define nominal value-added V as follows, where p_Y is the price of one unit of gross output, p_M is the price of one unit of raw materials, and all other variables have been defined above,

$$(1.1) \quad V(K,L,p_Y,p_M) \equiv \max_{Y,M}\{p_Y \cdot Y - p_M \cdot M : Y \le f(K,L,M)\}.$$

Now if p_Y and p_M vary in strict proportion, and we let $p \equiv a \cdot p_Y + b \cdot p_M$ be a price index in the prices p_Y and p_M with fixed weights a and b chosen so that p is positive for the base period, then it turns out that the deflated value-added function $v(K,L) \equiv V(K,L,p_Y,p_M)/p$ does not depend on the prices p_Y and p_M and moreover, the function v satisfies the usual neoclassical production function properties in K and L, if f

[4]The literature on this problem includes Geary (1944), Berlinguette and Leacy (1961), and David (1966).

[5]See Goldman and Uzawa (1964) and Blackorby, Primont, and Russell (1978) for a treatment of weak separability.

[6]Frequently, the separability assumption is implicitly or explicitly strengthened to read $Y = f(K,L,M) \equiv g(h(K,L),M)$ where g is a Leontief or fixed coefficients type production function. In this latter case, either Y, M or $Y - M$ may be used as an index of real value-added.

satisfies suitable regularity conditions. Thus if producers behave in a profit-maximizing manner, and if the prices of outputs and intermediate inputs vary in proportion, then the replacement of gross output Y by deflated value-added $v \equiv (p_Y \cdot Y - p_M \cdot M)/p$ can indeed be justified from the viewpoint of production function studies. This second method of justifying the substitution of real value-added for gross output is due to Khang (1971) and Bruno (Chapter III.1).

In Section 4 of the present chapter, we pursue the approach of Khang and Bruno, allowing for joint production and non-differentiable production functions. We simply note that a nominal or money value-added function [such as $V(K,L,p_Y,p_M)$ defined by equation (1.1)] is a special case of a *variable profit function*,[7] and thus known results may be used in order to characterize the properties of the value-added function with respect to prices and quantities. We then show that if all output prices and intermediate input prices vary in strict proportion, then the deflated value-added function has the properties of a neoclassical production function.

In Section 5, we consider the problem of minimizing the cost of producing one unit of nominal value-added and develop a duality theorem between the resulting cost function and the value-added function.[8]

In Section 6, we conclude by engaging in some armchair empiricism and we suggest that most postwar production function studies using deflated value added as output are probably somewhat biased.

Section 7 presents proofs for the longer theorems.

In the remainder of the present section, we review some mathematical theorems which will be utilized in the following sections of the chapter.

(1.2) *Definition.* A function f defined on S, a subset of Euclidean M space, is said to be *continuous from below* at a point $\mathbf{x}^0 \in S$ if for every $\epsilon > 0$, there exists a neighbourhood $U(\mathbf{x}^0)$ such that $\mathbf{x} \in U(\mathbf{x}^0)$ implies $f(\mathbf{x}) > f(\mathbf{x}^0) - \epsilon$.

(1.3) *Definition.* A numerical function f defined on S is said to be *continuous from above* at a point $\mathbf{x}^0 \in S$ if for every $\epsilon > 0$ there

[7]The variable profit function was introduced into the economics literature by Samuelson (1953–54, p. 20). Its properties were more formally studied by Gorman (1968a) who called it a gross profit function, Diewert (1973a), McFadden (Chapter I.1), who uses the terminology restricted profit function, and Lau (1976a).

[8]Denny (1972) has independently studied aspects of this duality relationship in the context of a non-constant returns to scale production function. Woodland (1977) has extensively applied this duality in the context of international trade theory.

exists a neighbourhood $U(\mathbf{x}^0)$ such that $\mathbf{x} \in U(\mathbf{x}^0)$ implies $f(\mathbf{x}) < f(\mathbf{x}^0) + \epsilon$.

Note that f is continuous at $\mathbf{x}^0$ if and only if it is continuous from below and above.[9]

We say that a function f is continuous from above over S if it is continuous from above at each point of S.

(1.4) *Lemma* [Berge (1963, p. 76), Rockafellar (1970, p. 51)]. The function f is continuous from above over S if and only if the set $\{\mathbf{x}: f(\mathbf{x}) \geq \alpha;\ \mathbf{x} \in S\}$ is closed in S for every scalar α.

(1.5) *Theorem* [Berge (1963, p. 76)]. If S is a compact (i.e., closed and bounded) subset of Euclidean M space, then a continuous from above function f attains in S the value $M \equiv \sup_{\mathbf{x}}\{f(\mathbf{x}): \mathbf{x} \in S\}$.

In the following definitions, let S denote a subset of R^M, T a subset of R^N, $\{\mathbf{x}^n\}$ a sequence of points of S and $\{\mathbf{y}^n\}$ a sequence of points of T.

(1.6) *Definition.* ϕ is a *correspondence* from S into T if for every $\mathbf{x} \in S$, there exists an image set $\phi(\mathbf{x})$ which is a subset of T.

(1.7) *Definitions.* A correspondence ϕ is *upper semicontinuous* at the point $\mathbf{x}^0 \in S$ if $\lim \mathbf{x}^n = \mathbf{x}^0$; $\mathbf{y}^n \in \phi(\mathbf{x}^n)$; $\lim \mathbf{y}^n = \mathbf{y}^0$ implies $\mathbf{y}^0 \in \phi(\mathbf{x}^0)$. A correspondence ϕ is *lower semicontinuous* at $\mathbf{x}^0 \in S$ if $\lim \mathbf{x}^n = \mathbf{x}^0$; $\mathbf{y}^0 \in \phi(\mathbf{x}^0)$ implies that there exists a sequence $\{\mathbf{y}^n\}$ such that $\mathbf{y}^n \in \phi(\mathbf{x}^n)$ and $\lim \mathbf{y}^n = \mathbf{y}^0$. A correspondence ϕ is continuous at $\mathbf{x}^0 \in S$ if it is both upper and lower semicontinuous at $\mathbf{x}^0$.

(1.8) *Upper Semicontinuity Maximum Theorem* [Berge (1963, p. 116)]. Let f be a continuous from above function defined for $(\mathbf{x},\mathbf{y})$ such that $\mathbf{x} \in S$ and $\mathbf{y} \in T$ where T is a compact subset of R^N. Suppose that ϕ is a correspondence from S into T and that ϕ is upper semicontinuous at $\mathbf{x}^0 \in S$. Then the function g defined by $g(\mathbf{x}) \equiv \max_{\mathbf{y}}\{f(\mathbf{x},\mathbf{y}): \mathbf{y} \in \phi(\mathbf{x})\}$ is continuous from above at $\mathbf{x}^0$.

(1.9) *Maximum Theorem* [Debreu (1952, pp. 889–890, and 1959, p. 19), Berge (1963, p. 116)]. Let f be a continuous real valued function defined for $(\mathbf{x},\mathbf{y})$ such that $\mathbf{x} \in S$ and $\mathbf{y} \in T$ where T is a compact subset of R^N. Suppose that ϕ is a correspondence from

[9]The property of continuity from above is often called upper semicontinuity [cf. Berge (1963, p. 74) or Rockafellar (1970, p. 51)]. However, we will use the term upper semicontinuity to describe a property of a *correspondence*.

S into T and that ϕ is continuous at $\mathbf{x}^0 \in S$. Define the maximum $g(\mathbf{x}^0) \equiv \max_{\mathbf{y}}\{f(\mathbf{x}^0,\mathbf{y}):\mathbf{y} \in \phi(\mathbf{x}^0)\}$ and the set of maximizers $\xi(\mathbf{x}^0) \equiv \{\mathbf{y}:\mathbf{y} \in \phi(\mathbf{x}^0)$ and $f(\mathbf{x}^0,\mathbf{y}) = g(\mathbf{x}^0)\}$. Then the function g is continuous at $\mathbf{x}^0$ and the correspondence ξ is upper semicontinuous at $\mathbf{x}^0$.

Now that we have disposed of the mathematical preliminaries, we are in a position to prove a very general version of Hicks' Aggregation Theorem.

2. Hicks' Aggregation Theorem in the Consumer Context

Consider the following micro or *disaggregated utility maximization problem*, where f is the consumer's utility function, $\mathbf{x} \geq \mathbf{0}_M$ is a non-negative M-dimensional vector of commodity rentals,[10] $\mathbf{y} \geq \mathbf{0}_N$ is a non-negative N-dimensional vector of commodity rentals, $\mathbf{w} \gg \mathbf{0}_M$ is an M-dimensional vector of positive rental prices, $\mathbf{p} \gg \mathbf{0}_N$ is an N-dimensional vector of positive rental prices and $Y \geq 0$ is the consumer's "income",

$$(2.1) \quad \max_{\mathbf{x},\mathbf{y}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \geq \mathbf{0}_M,\ \mathbf{y} \geq \mathbf{0}_N\,;\ \mathbf{w}^T\mathbf{x} + \mathbf{p}^T\mathbf{y} \leq Y\}.$$

Observe that if $\mathbf{w} \gg \mathbf{0}_M$, $\mathbf{p} \gg \mathbf{0}_N$, $Y \geq 0$ and the utility function f is continuous from above, then at least one solution $\mathbf{x}^*,\mathbf{y}^*$ to the disaggregated utility maximization problem 2.1 will exist, using Theorem 1.5. Thus in what follows, a minimum regularity condition we will impose on the micro utility function f is that it be continuous from above.

Suppose that the prices $\mathbf{w}$, which correspond to the first group of commodities, satisfy the relationship $\mathbf{w} = p_0\boldsymbol{\alpha}$ where $\boldsymbol{\alpha} \gg \mathbf{0}_M$ is a fixed vector of constants and $p_0 > 0$ is a scalar. In other words, from period to period, p_0 and $\mathbf{p}$ may vary in an arbitrary fashion, but the variation in the price vector $\mathbf{w}$ is limited by the equation $\mathbf{w} = p_0\boldsymbol{\alpha}$. This vector of fixed constants $\boldsymbol{\alpha}$ is used in order to define the following *aggregated utility function*:

$$(2.2) \quad U_\alpha(y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \geq \mathbf{0}_M,\ \boldsymbol{\alpha}^T\mathbf{x} \leq y_0\},$$

[10] *Notation:* $\mathbf{0}_M$ denotes an M-dimensional column vector of zeroes. $\mathbf{x}^T$ denotes the transpose of the column vector $\mathbf{x}$. $\mathbf{x} \geq \mathbf{0}_M$ means each component of $\mathbf{x}$ is non-negative, and $\mathbf{x} \gg \mathbf{0}_M$ means each component is positive.

where $\mathbf{y} \geq \mathbf{0}_N$, $y_0 \geq 0$, $\boldsymbol{\alpha} \gg \mathbf{0}_M$, and f is the micro utility function. We note that if the micro utility function is continuous from above, then Theorem 1.5 implies that the aggregated (over commodities) utility function defined by equation 2.2 will be well defined as a maximum since the set $\{\mathbf{x}:\mathbf{x} \geq \mathbf{0}_M, \boldsymbol{\alpha}^T\mathbf{x} \leq y_0\}$ is compact if $\boldsymbol{\alpha} \gg \mathbf{0}_M$ and $y_0 \geq 0$.

Now given the macro utility function U_α, $p_0 > 0$, $\mathbf{p} \gg \mathbf{0}_N$, $Y \geq 0$, the macro or *aggregated utility maximization problem* is defined as follows:

(2.3) $$\max_{y_0,\mathbf{y}}\{U_\alpha(y_0,\mathbf{y}):p_0y_0 + \mathbf{p}^T\mathbf{y} \leq Y;\, y_0 \geq 0, \mathbf{y} \geq \mathbf{0}_N\}.$$

(2.4) *Hicks' Aggregation Theorem* [Hicks (1946, pp. 312–313), Wold (1953, pp. 109–110), Gorman (1953, pp. 76–77)]. Let the micro utility function f be continuous from above, let $p_0 > 0$, $\boldsymbol{\alpha} \gg \mathbf{0}_M$, $\mathbf{w} = p_0\boldsymbol{\alpha}$, $\mathbf{p} \gg \mathbf{0}_N$ and $Y \geq 0$. Then (i) the macro utility function U_α defined by equation 2.2 is also continuous from above (and thus the macro utility maximization problem given by equation 2.3 has a solution), (ii) if $(\mathbf{x}^*,\mathbf{y}^*)$ is any solution to the disaggregated utility maximization problem 2.1, then $(y_0^*,\mathbf{y}^*)$ is a solution to the aggregated utility maximization problem 2.3 where $y_0^* \equiv \boldsymbol{\alpha}^T\mathbf{x}^* \equiv \mathbf{w}^T\mathbf{x}^*/p_0$ (and thus the usual procedure of defining an aggregate commodity as expenditure on a group of commodities divided by a price index is justified), and (iii) if the micro utility function f has any of the properties (a) to (e) below in addition to continuity from above, then the macro utility function U_α defined by equation 2.2 also has the corresponding properties: (a) local non-satiation,[11] (b) continuity, (c) non-decreasing[12] in its arguments, (d) quasi-concavity,[13] (e) strict quasi-concavity.

A proof of the above theorem is given in Section 7.[14]

We note that if a micro utility function f has the properties of continuity from above plus local non-satiation, then data generated as

[11]The utility function $f(\mathbf{z})$ where $\mathbf{z} \geq \mathbf{0}_{M+N}$ is subject to local non-satiation if for every $\mathbf{z}^0 \geq \mathbf{0}_{M+N}$ and $\delta > 0$, there exists $\mathbf{z} \geq \mathbf{0}_{M+N}$ such that $(\mathbf{z}-\mathbf{z}^0)^T(\mathbf{z}-\mathbf{z}^0) \leq \delta^2$ and $f(\mathbf{z}) > f(\mathbf{z}^0)$.

[12]The utility function f is non-decreasing if $\mathbf{0}_{M+N} \leq \mathbf{z}^1 \leq \mathbf{z}^2$ implies $f(\mathbf{z}^1) \leq f(\mathbf{z}^2)$.

[13]The utility function f is quasi-concave if and only if for every scalar k the set $L(k) \equiv \{\mathbf{z}:f(\mathbf{z}) \geq k;\, \mathbf{z} \geq \mathbf{0}_{M+N}\}$ is convex; f is strictly quasi-concave if and only if for every k, $L(k)$ is a strictly convex set. A function f is quasi-convex if and only if $-f$ is quasi-concave.

[14]One can also show, using some results due to Danskin (1967, p. 24), that if the micro utility function f has the properties of local non-satiation, strict quasi-concavity and in addition is twice continuously differentiable with respect to its arguments, then the macro utility function will also be twice continuously differentiable with respect to its arguments.

solutions to the micro utility maximization problem 2.1 will satisfy the strong axiom of revealed preference.[15] Hence Theorem 2.4 implies that the aggregated data will also satisfy the strong axiom of revealed preference, provided that prices of the goods in the aggregate vary in strict proportion over time. Thus if the last condition is satisfied, the use of an aggregate commodity in place of the micro commodities can be justified from the viewpoint of the received theory of consumer demand.

If the vector $\boldsymbol{\alpha}$ does not remain constant over time, then the macro utility function $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ defined by equation 2.2 will be a function of $\boldsymbol{\alpha}$. In the following section, we determine the properties of U with respect to $\boldsymbol{\alpha}$.

3. Duality between Direct and Conditional Indirect Utility Functions

We now allow the vector of parameters α found in Definition 2.2 to vary and we determine the properties of the macro utility function $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ with respect to $\boldsymbol{\alpha}$.

(3.1) *Theorem.* Let the micro utility function $f(\mathbf{x},\mathbf{y})$ be continuous from above with respect to $\mathbf{x} > \mathbf{0}_M$ for a fixed $\mathbf{y} \geq \mathbf{0}_N$ and let $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ be defined by equation 2.2 for $\boldsymbol{\alpha} \gg \mathbf{0}_M$, $y_0 \geq 0$, $\mathbf{y} \geq \mathbf{0}_N$. Then for fixed y_0 and $\mathbf{y}$, $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is (i) continuous from above with respect to $\boldsymbol{\alpha}$, (ii) non-increasing with respect to $\boldsymbol{\alpha}$, and (iii) a quasi-convex function of $\boldsymbol{\alpha}$ over the set $S \equiv \{\boldsymbol{\alpha}:\boldsymbol{\alpha} \gg \mathbf{0}_M\}$. (iv) For fixed $\mathbf{y}$, $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is homogeneous of degree zero in $(\boldsymbol{\alpha},y_0)$, i.e., if $\boldsymbol{\alpha} \gg \mathbf{0}_M$, $\lambda > 0$, $y_0 \geq 0$, then $U_{\lambda\boldsymbol{\alpha}}(\lambda y_0,\mathbf{y}) = U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$. (v) If the micro utility function f is a continuous function over the non-negative $(M+N)$-dimensional orthant, then $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is jointly continuous with respect to $\boldsymbol{\alpha} \gg \mathbf{0}_M$, $y_0 \geq 0$, and $\mathbf{y} \geq \mathbf{0}_N$.

A proof of the above theorem is given in Section 7. Note that Theorem 3.1 yields the properties of the *indirect utility function* as a special case (i.e., let $N = 0$ and the vector $\mathbf{y}$ vanishes from Definition 2.2, and then y_0 may be interpreted as "income") and thus Theorem 3.1 generalizes somewhat some aspects of the duality theorems between

[15]See Houthakker (1950, p. 163) for a statement of the strong axiom of revealed preference.

direct and indirect utility (or production) functions due to Newman (1965, pp. 138–172), Lau (1969a), Weddepohl (1970, p. 125) and Shephard (1970, pp. 13–23, 105–111, 301–305).[16] If we look at $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ as a function of $\boldsymbol{\alpha}$, Theorem 3.1 tells us that the set of "prices" $\{\boldsymbol{\alpha}: U_{\boldsymbol{\alpha}}(y_0,\mathbf{y}) \leq k;\ \boldsymbol{\alpha} \gg \mathbf{0}_M\}$ will be a convex, closed (in the positive orthant), "non-backward-bending" set, provided only that the micro utility function is continuous; i.e., the set $\{\boldsymbol{\alpha}: U_{\boldsymbol{\alpha}}(y_0,\mathbf{y}) \leq k;\ \boldsymbol{\alpha} \gg \mathbf{0}_M\}$ will look like an ordinary indifferent-or-preferred-to set. However, note that as k increases, the set $\{\boldsymbol{\alpha}: U_{\boldsymbol{\alpha}}(y_0,\mathbf{y}) \leq k;\ \boldsymbol{\alpha} \gg \mathbf{0}_M\}$ will generally move *towards* the origin instead of away from it; i.e., as the "prices" $\boldsymbol{\alpha}$ become smaller, the set of feasible x's in Definition 2.2 becomes larger and thus we would expect $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ to increase.

The macro utility function $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is called a *conditional indirect utility function* by Pollak (1969), or a *variable indirect utility function* by Epstein (1975). Epstein also showed that the properties of the macro utility function which occurred in Theorems 2.4 and 3.1 completely characterize a certain class of preferences. That is, suppose the direct utility function $f(\mathbf{x},\mathbf{y})$ is a continuous, non-decreasing, quasi-concave function of $(\mathbf{x},\mathbf{y})$ over $\mathbf{x} \geq \mathbf{0}_M$, $\mathbf{y} \geq \mathbf{0}_N$ and the conditional indirect utility function $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is defined by equation 2.2. Then $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ is: (i) a finite continuous real valued function over the set $S \equiv \{(\boldsymbol{\alpha},y_0,\mathbf{y}): \boldsymbol{\alpha} \gg \mathbf{0}_M,\ y_0 \geq 0,\ \mathbf{y} \geq \mathbf{0}_N\}$, (ii) non-increasing and quasi-convex in $\boldsymbol{\alpha} \gg \mathbf{0}_M$ for every $y_0,\mathbf{y}$, (iii) homogeneous of degree zero in $(\boldsymbol{\alpha},y_0)$ for every $\mathbf{y}$, and (iv) non-decreasing and quasi-concave in $(y_0,\mathbf{y})$ for every $\boldsymbol{\alpha}$. Now extend the domain of definition of $U_{\boldsymbol{\alpha}}(y_0,\mathbf{y})$ to $\boldsymbol{\alpha} \geq \mathbf{0}_M$ by continuity. (The resulting function need not be finite if any component of $\boldsymbol{\alpha}$ is zero.) Define the direct utility function f^* using the conditional indirect utility function $U_{\boldsymbol{\alpha}}$ for $\mathbf{x} \geq \mathbf{0}_M$ and $\mathbf{y} \gg \mathbf{0}_N$ as follows:

$$(3.2)\quad f^*(\mathbf{x},\mathbf{y}) \equiv \min_{\boldsymbol{\alpha}}\{U_{\boldsymbol{\alpha}}(1,\mathbf{y}): \boldsymbol{\alpha}^T x \leq 1,\ \boldsymbol{\alpha} \geq \mathbf{0}_M\}.$$

Then Epstein shows that $f = f^*$; i.e., the conditional indirect utility

[16]The main difference between Theorem 3.1 and the results of Newman, Lau and Shephard lies in the extremely weak regularity conditions we have imposed on the direct utility function f. However, there are other minor differences. Newman (1965, p. 160) appears to have the curvature of the level sets of the indirect utility function going the wrong way, Lau (1969a, p. 376) asserts that the indirect utility function is a convex function in prices rather than a quasi-convex function and Shephard (1970, p. 106) is able to prove that the indirect utility function is continuous from below (rather than being continuous from above). However, Shephard's result can be traced to the fact that he defines the indirect utility functions as a sup rather than as a maximum.

function U_α completely characterizes the direct utility function f.[17] For applications and further extensions of this duality between conditional indirect and direct utility functions, see Blackorby, Primont and Russell (1977b, 1978).

As Hanoch has noted in Chapter I.2, in the production theory context, the analogue of the indirect utility function is the indirect production function. Thus we analogously define the conditional indirect production function (while introducing a more traditional notation) as

$$(3.3)\quad g(\mathbf{v},\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{v}^T\mathbf{x} \leq 1, \mathbf{x} \geq \mathbf{0}_M\},$$

where f is the direct production function, $\mathbf{x}$ is an M-dimensional vector of variable inputs, $\mathbf{y} \geq \mathbf{0}_N$ is an N-dimensional vector of fixed inputs, $\mathbf{v} \equiv \mathbf{p}/Y$ is a vector of *normalized* prices, $\mathbf{p} \gg \mathbf{0}_M$ is a vector of variable input rental prices, while $Y > 0$ is expenditure on the variable inputs, and g is the conditional indirect production function. Thus $g(\mathbf{v},\mathbf{y})$ is the analogue to $U_v(1,\mathbf{y})$ and the same duality theorem as held between $f(\mathbf{x},\mathbf{y})$ and $U_\alpha(1,\mathbf{y})$ will hold between f and g, except that it is natural to assume that output be positive if all inputs are positive which in turn will imply that $g(\mathbf{v},\mathbf{y})$ is positive for positive arguments. We denote output as $u \equiv f(\mathbf{x},\mathbf{y})$, since later in this section, we will interpret u as utility and f and g as direct and conditional indirect utility functions, respectively.

Looking at equation 3.3, we see that the conditional indirect production function $g(\mathbf{v},\mathbf{y})$ gives the solution to the problem of maximizing output $u = f(\mathbf{x},\mathbf{y})$ given an expenditure constraint on variable inputs $\mathbf{x}$ of the form $\mathbf{p}^T\mathbf{x} \leq Y$ (or $\mathbf{v}^T\mathbf{x} \leq 1$ where $\mathbf{v} \equiv \mathbf{p}/Y$) and given the vector of fixed inputs $\mathbf{y}$. On the other hand, given a conditional indirect production function g satisfying the appropriate regularity conditions, the corresponding direct production function f can be calculated as the solution to the following minimization problem (which is the counterpart to equation 3.2):

$$(3.4)\quad f(\mathbf{x},\mathbf{y}) \equiv \min_{\mathbf{v}}\{g(\mathbf{v},\mathbf{y}):\mathbf{v}^T\mathbf{x} \leq 1, \mathbf{v} \geq \mathbf{0}_M\}.$$

In the theory of production, it is often assumed that the production function f is positively, linearly homogeneous; i.e., for every $\mathbf{x} \geq \mathbf{0}_M$, $\mathbf{y} \geq \mathbf{0}_N$ and scalar $\lambda > 0$, $f(\lambda\mathbf{x},\lambda\mathbf{y}) = \lambda f(\mathbf{x},\mathbf{y})$. The following theorem noted

[17]Epstein also notes that the continuity problems which were discussed in Diewert (1974a, pp. 121–123) also occur in the present context but that the same techniques that were used in Diewert can be used in the present context.

by Epstein (1973b) indicates the implications of linear homogeneity of f on g.

(3.5) *Theorem* [Epstein (1973b)]. Let the direct production function f and its dual conditional indirect production function g satisfy Epstein's regularity conditions. Then f is positively linearly homogeneous if and only if for every $\mathbf{v} \gg \mathbf{0}_M$, $\mathbf{y} \geq \mathbf{0}_N$, $\lambda > 0$, we have

$$(3.6) \quad g(\lambda^{-1}\mathbf{v},\lambda\mathbf{y}) = \lambda g(\mathbf{v},\mathbf{y}).$$

If we define a transformed conditional indirect production function g^* for $\mathbf{v} \gg \mathbf{0}_M$ by $g^*(\mathbf{v}^{-1},\mathbf{y}) \equiv g(\mathbf{v},\mathbf{y})$, where $\mathbf{v}^{-1} \equiv (v_1^{-1},v_2^{-1},\ldots,v_M^{-1})$ is a vector which has as components the reciprocals of the components of $\mathbf{v} \equiv (v_1,v_2,\ldots,v_M)$, then the thrust of Theorem 3.5 is that the direct production function f is positively linearly homogeneous if and only if g^* is positively linearly homogeneous in its arguments.

Just as linear homogeneity places certain restrictions on the first- and second-order partial derivatives of a direct production function, the homogeneity property defined by equation 3.6 places restrictions on the first- and second-order partial derivatives of g. Assuming that g is twice continuously differentiable at a point $\mathbf{v}^* \gg \mathbf{0}_M$, $\mathbf{y}^* \gg \mathbf{0}_N$, if we partially differentiate equation 3.6 with respect to λ and set $\lambda = 1$, we obtain the following restriction on the first-order partials of g:

$$(3.7) \quad -\mathbf{v}^{*T}\nabla_{\mathbf{v}}g(\mathbf{v}^*,\mathbf{y}^*) + \mathbf{y}^{*T}\nabla_{\mathbf{y}}g(\mathbf{v}^*,\mathbf{y}^*) = g(\mathbf{v}^*,\mathbf{y}^*),$$

where $\nabla_{\mathbf{v}}g(\mathbf{v}^*,\mathbf{y}^*)$ denotes the vector of first-order partial derivatives of g with respect to $\mathbf{v}$ evaluated at $\mathbf{v}^*$, $\mathbf{y}^*$ and $\nabla_{\mathbf{y}}g(\mathbf{v}^*,\mathbf{y}^*)$ denotes the partial derivatives of g with respect to the components of $\mathbf{y}$.

On the other hand, if we partially differentiate the identity equation 3.6 with respect to the components of $\mathbf{v}$ and then partially differentiate the resulting M equations with respect to λ, upon setting $\lambda = 1$, we obtain the following M equations:

$$(3.8) \quad -\nabla^2_{\mathbf{vv}}g(\mathbf{v}^*,\mathbf{y}^*)\mathbf{v}^* + \nabla^2_{\mathbf{vy}}g(\mathbf{v}^*,\mathbf{y}^*)\mathbf{y}^* = 2\nabla_{\mathbf{v}}g(\mathbf{v}^*,\mathbf{y}^*),$$

where $\nabla^2_{\mathbf{vv}}g(\mathbf{v}^*,\mathbf{y}^*)$ denotes the M by M matrix of second-order partial derivatives of g with respect to $\mathbf{v}$ while $\nabla^2_{\mathbf{vy}}g(\mathbf{v}^*,\mathbf{y}^*)$ denotes the M by N matrix of second-order partial derivatives of g with respect to the components of $\mathbf{v}$ and $\mathbf{y}$.

Finally, if we partially differentiate equation 3.6 with respect to the components of y, then partially differentiate the resulting N equations with respect to λ, and then set $\lambda = 1$, we obtain the following N equations:

$$(3.9) \quad -\nabla^2_{yv}g(\mathbf{v}^*,\mathbf{y}^*)\mathbf{v}^* + \nabla^2_{yy}g(\mathbf{v}^*,\mathbf{y}^*)\mathbf{y}^* = \mathbf{0}_N.$$

Since g is assumed to be twice continuously differentiable at $(\mathbf{v}^*,\mathbf{y}^*)$, the matrices of second-order partial derivatives $\nabla^2_{vv}g(\mathbf{v}^*,\mathbf{y}^*)$ and $\nabla^2_{yy}g(\mathbf{v}^*,\mathbf{y}^*)$ will be symmetric. Thus these symmetry conditions and the $(1+M+N)$ restrictions 3.7 to 3.9 imply that there are $M(M-1)/2 + N(N-1)/2 + (1+M+N)$ linear restrictions on the first- and second-order partial derivatives of g (and the level of g), evaluated at the point $(\mathbf{v}^*,\mathbf{y}^*)$. Thus it can be seen that the following functional form for a conditional indirect production function due to Epstein (1973b) can provide a second-order differential[18] approximation to an arbitrary twice continuously differentiable conditional indirect production function which has the homogeneity property 3.6[19] (and thus corresponds to a constant returns direct production function):

$$(3.10) \quad u = g(\mathbf{v},\mathbf{y}) \equiv \sum_{m=1}^{M}\sum_{k=1}^{M} a_{mk}v_m^{-1/2}v_k^{-1/2} + 2\sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}v_m^{-1/2}y_n^{1/2} + \sum_{n=1}^{N}\sum_{r=1}^{N} b_{nr}y_n^{1/2}y_r^{1/2},$$

where $a_{mk} = a_{km}$ for every m, k, and $b_{nr} = b_{rn}$ for every n,r. It is easy to verify that the functions $y_n^{1/2}$ and $y_n^{1/2}y_r^{1/2}$ are concave in y and that the functions $v_m^{-1/2}$ and $v_m^{-1/2}v_k^{-1/2}$ are convex in v.[20] Thus if all of the coefficients a_{mk}, b_{nr} and c_{mn} are non-negative, it can be verified that g defined by equation 3.10 is (i) non-negative, (ii) non-increasing and convex (and hence quasi-convex) in v for fixed y, (iii) non-decreasing and concave (and hence quasi-concave) in y for fixed v, and (iv) g satisfies the homogeneity property 3.6. Thus g is dual to a well-behaved linearly homogeneous direct production function.[21]

[18]The term is due to Lau (1974), but the concept is discussed by McFadden (Chapter II.2) and is termed the second-order approximation property.

[19]It is easier to see that the functional form defined by equation 3.10 provides a second-order differential approximation to a twice continuously differentiable transformed conditional indirect production function, $g^*(v^{-1},y) = g^*(z)$, since $g(z)$ defined by equation 3.10 is a Generalized Leontief functional form in $z \equiv (v^{-1},y)$ and thus the "flexibility" theorem of Diewert (1974a, p. 117) applies.

[20]It is well-known that the reciprocal of a positive concave function is convex; e.g., see Zangwill (1969, p. 60).

[21]If some of the coefficients of g defined by equation 3.10 are negative, then g can still satisfy the appropriate regularity conditions over a region and be a valid representation of technology over that region. See Epstein (1973b) and Diewert (1973a, pp. 305–306).

Roy's Identity[22] may be used in order to generate the firm's system of M *conditional* or *short-run demand functions* $x(\mathbf{v}^*,\mathbf{y}^*)$ at any point $(\mathbf{v}^*,\mathbf{y}^*)$ where the conditional indirect production function is differentiable,

$$(3.11)\quad \mathbf{x}(\mathbf{v}^*,\mathbf{y}^*) \equiv \nabla_v g(\mathbf{v}^*,\mathbf{y}^*)/\mathbf{v}^{*T}\nabla_v g(\mathbf{v}^*,\mathbf{y}^*).$$

If the firm's conditional indirect production function is defined by equation 3.10, then the firm's output maximizing demands $\mathbf{x} \equiv (x_1,x_2,\ldots,x_M)$ for variable inputs given the expenditure constraint $\mathbf{v}^T\mathbf{x} \leq 1$ are defined by the following equation (upon application of Roy's Identity 3.11):

$$(3.12)\quad x_m = \frac{-\sum_{k=1}^{M} a_{mk} v_m^{-3/2} v_k^{-1/2} - \sum_{n=1}^{N} c_{mn} v_m^{-3/2} y_n^{1/2}}{-\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij} v_i^{-1/2} v_j^{-1/2} - \sum_{h=1}^{M}\sum_{k=1}^{N} c_{hk} v_h^{-1/2} y_k^{1/2}},$$

for $m = 1,2,\ldots,M$.

Given data on output u, variable inputs $\mathbf{x}$, fixed inputs $\mathbf{y}$ and variable input prices $\mathbf{p}$ (so that normalized variable input prices $\mathbf{v} \equiv \mathbf{p}/\mathbf{p}^T\mathbf{x}$ can be calculated), we may use equation 3.10 and the M variable input demand equations 3.12 (only $M-1$ of them are independent) in order to estimate the parameters of technology, a_{mk}, b_{nr} and c_{mn}.

Suppose now that the firm can periodically vary its fixed inputs y. For example, we might suppose that output and the short-run inputs are chosen every week, but that once a year long-run inputs are also changed. Suppose that the firm faces long-run input rental prices $\mathbf{q} = (q_1,q_2,\ldots,q_N)$ and that it wishes to spend no more than $E > 0$ dollars on fixed inputs during the week when long-run inputs are chosen. Then given short-run input prices $\mathbf{p}$ and a budget of $Y > 0$ dollars to spend on short-run inputs, the firm will wish to choose the vector of long-run inputs y which solves the following maximization problem:

$$(3.13)\quad \max_{\mathbf{y}}\{g(\mathbf{p}/Y,\mathbf{y}):\mathbf{q}^T\mathbf{y} \leq E, \mathbf{y} \geq \mathbf{0}_N\} = \max_{\mathbf{y}}\{g(\mathbf{v};\mathbf{y}):\mathbf{w}^T\mathbf{y} \leq 1, \mathbf{y} \geq \mathbf{0}_N\},$$

where $\mathbf{v} \equiv \mathbf{p}/Y$ is the vector of short-run normalized prices, $\mathbf{w} \equiv \mathbf{q}/E$ is the vector of long-run normalized input prices, and g is the producer's conditional indirect production function.

The producer's system of long-run *inverse demand* functions $\mathbf{w}(\mathbf{v}^*,\mathbf{y}^*)$

[22]See Roy (1942), Konyus and Byushgens (1926), or Diewert (1974a, p. 124).

can be defined as the set of normalized long-run prices such that $\mathbf{y}^* \geq \mathbf{0}_N$ is a solution to the constrained output maximization problem 2.13 when $\mathbf{v} = \mathbf{v}^*$ and $\mathbf{w} = \mathbf{w}(\mathbf{v}^*,\mathbf{y}^*)$. If the conditional indirect production function g is non-decreasing, quasi-concave and once differentiable in $\mathbf{y}$, then the producer's system of long-run inverse demand functions can be calculated by using Wold's Identity,[23]

$$(3.14) \quad \mathbf{w}(\mathbf{v}^*,\mathbf{y}^*) \equiv \nabla_{\mathbf{y}} g(\mathbf{v}^*,\mathbf{y}^*)/\mathbf{y}^{*T}\nabla_{\mathbf{y}} g(\mathbf{v}^*,\mathbf{y}^*).$$

If the firm's conditional indirect production function is defined by equation 3.10, then application of Wold's Identity yields the following system of long-run inverse demand functions:

$$(3.15) \quad \frac{q_n}{E} \equiv w_n = \frac{\sum_{r=1}^{N} b_{nr} y_n^{-1/2} y_r^{1/2} + \sum_{m=1}^{M} c_{mn} v_m^{-1/2} y_n^{-1/2}}{\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij} y_i^{1/2} y_j^{1/2} + \sum_{h=1}^{M}\sum_{k=1}^{N} c_{hk} v_h^{-1/2} y_k^{1/2}}.$$

Given data on output u, variable inputs $\mathbf{x}$, "fixed" inputs $\mathbf{y}$, variable input prices $\mathbf{p}$ (recall $\mathbf{v} \equiv \mathbf{p}/\mathbf{p}^T\mathbf{x}$) and "fixed" input prices $\mathbf{q}$ (recall $\mathbf{w} \equiv \mathbf{q}/\mathbf{q}^T\mathbf{y}$), then the "weekly" equations 3.10 and 3.12 along with the N "annual" equations 3.14 (only $N-1$ of them are independent) can be used in order to estimate the unknown parameters a_{mk}, b_{nr}, and c_{mn}.

The idea of producers maximizing output subject to one or two expenditure constraints on inputs is perhaps not intuitively appealing.[24] It is more natural to think of producers as maximizing short-run profits with respect to variable outputs and inputs and then choosing fixed inputs to maximize long-run profits. In this case, the natural way to describe technology is by means of a variable profit function (a concept which will be discussed in the following section).[25]

[23]See Konyus and Byushgens (1926), Hotelling (1935), and Wold (1944, pp. 69–71) for proofs in the context of consumer theory. The proof consists of setting up the Lagrangean for the constrained maximization problem 3.13, partially differentiating and then eliminating the Lagrange multiplier.

[24]It is, however, logically consistent with competitive profit maximization, since the producer will always want to produce a given amount of output at minimum short- (and long-) run cost.

[25]Moreover, the use of the variable profit function instead of the conditional indirect production function can lead to a system of short-run input demand and output supply functions which are *linear* in the parameters of technology (instead of a nonlinear system like 3.12) if one uses an appropriate functional form and Hotelling's lemma as in Diewert (1974a, pp. 137–139). Furthermore, use of the variable profit function can lead to systems of long-run inverse demand functions which are also linear in the parameters of technology. See Diewert (1974a, p. 140).

However, the real usefulness of the concept of the conditional indirect production function lies not so much in the producer context but in the consumer context where the variable profit concept cannot be used – interpret output u as utility, $\mathbf{x}$ and $\mathbf{y}$ as short- and long-run commodity rentals with $\mathbf{p}$ and $\mathbf{q}$ being the corresponding vectors of rental prices, f as the direct utility function and g as the consumer's conditional indirect utility function. The main difference between the consumer and producer interpretations of the basic model is that while output can be observed utility cannot be observed. Thus in the consumer context, we cannot use equation 3.10 to aid us in the econometric estimation of the unknown parameters of the consumer's preferences.

Moreover since equations 3.12 and 3.15 are homogeneous of degree zero in the unknown parameters, in the consumer context when equation 3.10 is unavailable for econometric purposes, it will be necessary to make a normalization on the parameters (in order to determine the scale for utility) such as

$$(3.16) \quad \sum_{m=1}^{M} \sum_{k=1}^{M} a_{mk} + 2 \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} + \sum_{n=1}^{N} \sum_{r=1}^{N} b_{nr} = 1.$$

If units of measurement are chosen so that $v_m = 1$, $m = 1,2,\ldots,M$, and $y_n = 1$, $n = 1,2,\ldots,N$, during a base period, then the effect of the normalization 3.16 is to make utility equal to unity during the base period.

For an application of the above functional form to the problem of estimating a consumer's preferences under uncertainty, see Epstein (1973b, pp. 12–18). The above model assumes that the underlying direct production or utility function is linearly homogeneous. This restriction can be relaxed by introducing an additional fixed good, y_0 say, which always takes the value unity. This is equivalent to adding terms of the form $2\sum_{m=1}^{M} c_{m0} v_m^{-1/2} + b_{00} + 2\sum_{n=1}^{N} b_{n0} y_n^{1/2}$ to the conditional indirect function defined by equation 3.10.[26] We leave it to the reader to use Roy's Identity 3.11 and Wold's Identity 3.14 in order to derive the resulting short-run and long-run inverse demand functions. The parameter b_{00} will not be identifiable in the consumer context.

Let us now suppose that the consumer[27] has preferences defined over

[26]The resulting conditional indirect function generally has the second-order approximation property.

[27]A similar analysis can be applied in the producer context (toll-free roads come to mind as an obvious public good in the producer context), but again it is easier from an econometric point of view to phrase the analysis in terms of the variable profit function (where the fixed inputs include the public goods) in place of the conditional indirect production function.

market goods **x** and *public goods* **y**. The above model can be applied except that the consumer cannot generally optimize his choice of public good inputs, so that the maximization problem 3.13 is not applicable in the present context, and thus we cannot derive equations 3.15 to aid in the econometric estimation of the unknown parameters of preferences. We will be left with only equations 3.12 (or their counterparts in the case of nonhomothetic preferences) and thus the parameters b_{nr} which occur in equation 3.10 cannot be identified econometrically.

The following functional form for a conditional indirect utility function leads to a system of short-run derived demand functions which does not suffer from the above lack of identifiability problem:

$$(3.17) \quad g(\mathbf{v},\mathbf{y}) \equiv 2\sum_{m=1}^{M}\sum_{k=1}^{M} a_{mk}v_m^{-1/4}v_k^{-1/4}\left(\sum_{n=1}^{N} y_n^{1/2}\right) + 2\sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}v_m^{-1/2}y_n^{1/2}$$
$$+ 2\sum_{n=1}^{N}\sum_{r=1}^{N} b_{nr}y_n^{1/4}y_r^{1/4}\left(\sum_{m=1}^{M} v_m^{-1/2}\right) + 4\sum_{m=1}^{M} a_{0m}v_m^{-1/4}\left(\sum_{n=1}^{N} y_n^{3/4}\right)$$
$$+ 4\sum_{n=1}^{N} b_{0n}y_n^{1/4}\left(\sum_{m=1}^{M} v_m^{-3/4}\right) + a_0 + 4\sum_{m=1}^{M} a_m v_m^{-1/4}\left(\sum_{n=1}^{N} y_n^{1/4}\right)$$
$$+ 4\sum_{n=1}^{N} b_n y_n^{1/4}\left(\sum_{m=1}^{M} v_m^{-1/4}\right),$$

where $a_{mk} = a_{km}$, $b_{nr} = b_{rn}$, $a_{mm} \equiv 0$ for $m = 1,2,\ldots,M$, and $b_{nn} = 0$ for $n = 1,2,\ldots,N$. These last two sets of restrictions are necessary in order to identify the parameters c_{mn}. If all of the parameters are non-negative, then it is easy to verify that g defined by equation 3.17 satisfies the Diewert–Epstein regularity conditions for a conditional indirect function globally. The function defined by equation 3.17 generally has the second-order approximation property to an arbitrary twice differentiable conditional indirect function. Moreover, if the parameters a_m, $m = 0,1,\ldots,M$, and b_n, $n = 1,2,\ldots,N$, are set equal to zero, the resulting function has the homogeneity property 2.6 and can generally[28] differentially approximate an arbitrary twice differentiable conditional indirect function which corresponds to a linearly homogeneous direct function.

We leave to the reader the task of applying Roy's Identity 3.11 to the function defined by equation 3.17 in order to derive the variable input demand functions $\mathbf{x}(\mathbf{v},\mathbf{y})$. If we make a normalization on the parameters of equation 3.17 which makes utility equal to unity for a base obser-

[28]The proof is similar to Lemma 5.9 in Diewert (1973a). We require that a certain matrix whose coefficients are functions of the initial normalized prices and fixed quantities be nonsingular.

vation (and set $a_0 \equiv 0$), then it will turn out that the remaining parameters of g can be econometrically estimated using just the variable input demand functions $\mathbf{x}(\mathbf{v},\mathbf{y})$. Thus a consumer's preferences over market goods $\mathbf{x}$ and public goods $\mathbf{y}$ can be identified econometrically using only the market goods demand functions, provided that the consumer's preferences can be adequately represented by a conditional indirect utility function of the type defined by equation 3.17.

We conclude this section by discussing the implications of concavity of the direct utility or production function (as opposed to the weaker property of quasi-concavity) on the conditional indirect function, and the implications of convexity in normalized prices of the conditional indirect function (as opposed to the weaker property of quasi-convexity) on the direct function.[29]

(3.18) *Theorem.* Let f be a continuous, non-decreasing, quasi-concave direct function of $(\mathbf{x},\mathbf{y})$ over the non-negative orthant in $M+N$ space, and let $g(\mathbf{v},\mathbf{y})$ be the corresponding dual conditional indirect function defined by equation 3.3. Then $f(\mathbf{x},\mathbf{y})$ is concave in $(\mathbf{x},\mathbf{y})$ if and only if $g(\mathbf{p}/Y,\mathbf{y})$ is concave in $(Y,\mathbf{y})$ for every $\mathbf{p} \gg \mathbf{0}_M$. Moreover, under the above conditions, the conditional or restricted[30] cost function defined by

$$(3.19) \quad C(u;\mathbf{p},\mathbf{y}) \equiv \min_{\mathbf{x}}\{\mathbf{p}^T\mathbf{x} : f(\mathbf{x},\mathbf{y}) \geq u, \mathbf{x} \geq \mathbf{0}_M\}$$

is convex in $(u,\mathbf{y})$ for every $\mathbf{p} \gg \mathbf{0}_M$.

A proof of this theorem can be found in the appendix. The same proof with minor modifications can be used to show that under the above conditions on f, $f(\mathbf{x},\mathbf{y})$ is concave in $\mathbf{x}$ for a fixed $\mathbf{y}$ if and only if $g(\mathbf{p}/Y,\mathbf{y})$ is concave in Y for every $\mathbf{p} \gg \mathbf{0}_M$ or if and only if $C(u;\mathbf{p},\mathbf{y})$ is convex in u for every $\mathbf{p} \gg \mathbf{0}_M$.

(3.20) *Theorem.* Let f be a continuous, non-decreasing, quasi-concave direct function of $(\mathbf{x},\mathbf{y})$, and let g be the corresponding dual conditional indirect function. Then $g(\mathbf{v},\mathbf{y})$ is convex in normal-

[29]As well as being of direct interest in the production theory context where it is natural to assume concavity of the production function, these implications play a role in the analysis of consumer behavior under uncertainty; see Stiglitz (1969), Epstein (1973a and 1975), and Hanoch (1977).

[30]See McFadden (Chapter I.1) for a detailed analysis.

ized prices $v \gg 0_M$ for a fixed $y \geq 0_N$ if and only if $f(x/Z,y)$ is a convex function of the scalar variable $Z > 0$ for every $x \gg 0_M$.

A very simple proof of this theorem (due to L. Epstein) can be found in Section 7. If y is taken to be the empty set, then the above two theorems specialize to theorems about the direct and indirect production or utility functions, which have been studied elsewhere in this volume.

We now turn our attention to the theory of production and develop an analogue to Hicks' Aggregation Theorem in the context of value-added production functions.

4. Hicks' Aggregation Theorem in the Producer Context

Denote outputs by positive and inputs by negative numbers. Let T be the production possibilities set for a firm and let it be a non-empty subset of $(I+J)$-dimensional Euclidean space. We assume that there are I produced goods in the economy and that the firm under consideration produces some of these goods and uses others (such as raw materials, fuel and electricity) as intermediate inputs. We assume that there are J non-produced or primary factors of production (such as land, different grades of labour and capital services). We suppose that T satisfies the following regularity conditions:

(4.1) *Conditions I* on the Production Possibilities Set T:

(i) T is a closed, non-empty subset of $I+J$ dimensional space.

(ii) If $(u;v) \in T$, then $v \leq 0_J$ (last J goods are always inputs).

(iii) T is a convex set (non-increasing marginal rates of transformation).

(iv) T is a cone, i.e., $z \in T$, $\lambda \geq 0$ implies $\lambda z \in T$ (constant returns).

(v) If $z' \in T$, $z'' \leq z'$, then $z'' \in T$ (free disposal).

(vi) If $(u;v) \in T$, then the components of u are bounded from above (for finite fixed inputs, the set of producible outputs is also finite).

If $v \leq 0_J$ is a vector of fixed inputs and $p \gg 0_I$ is a vector of postive produced goods prices, we may define the producer's *variable profit* Π as

$$(4.2) \quad \Pi(p;v) \equiv \max_u \{p^T u : (u;v) \in T\}.$$

We can extend the domain of definition of the variable profit function Π to $\mathbf{p} \geq \mathbf{0}_I$ by continuity. Let us change the non-positive vector of primary inputs $\mathbf{v}$ into a non-negative vector of primary inputs by defining $\mathbf{x} \equiv -\mathbf{v}$. Now for $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{x} \geq \mathbf{0}_J$, we may define the *value-added function* V as

(4.3) $V(\mathbf{p};\mathbf{x}) \equiv \Pi(\mathbf{p};-\mathbf{x}), \quad \mathbf{p} \geq \mathbf{0}_I, \quad \mathbf{x} \geq \mathbf{0}_J.$

If the production possibilities set $\boldsymbol{T}$ satisfies Conditions I, it can be shown[31] that V satisfies the following conditions:

(4.4) *Conditions II* on the Value-Added Function V:

- (i) V is a *non-negative*, real valued function defined for all $(\mathbf{p};\mathbf{x}) \geq \mathbf{0}_{I+J}$.
- (ii) V is *linearly homogeneous* in $\mathbf{p}$ for fixed $\mathbf{x}$, i.e., if $\lambda > 0$, $V(\lambda\mathbf{p};\mathbf{x}) = \lambda V(\mathbf{p};\mathbf{x})$.
- (iii) V is a *continuous, convex* function in $\mathbf{p}$ for fixed $\mathbf{x}$.
- (iv) V is *linearly homogeneous* in $\mathbf{x}$ for fixed $\mathbf{p}$, i.e., if $\lambda > 0$, $V(\mathbf{p};\lambda\mathbf{x}) = \lambda V(\mathbf{p};\mathbf{x})$.
- (v) V is *non-decreasing* in $\mathbf{x}$ for fixed $\mathbf{p}$.
- (vi) V is a *continuous, concave* function in $\mathbf{x}$ for fixed $\mathbf{p}$.

On the other hand, given a value-added function V satisfying Conditions II above, we may obtain the production possibilities set $\boldsymbol{T}$ satisfying Conditions I[32] which corresponds to V by means of the following definition:

(4.5) $\boldsymbol{T} \equiv \{(\mathbf{u};-\mathbf{x}):\mathbf{p}^T\mathbf{u} \leq V(\mathbf{p};\mathbf{x}) \text{ for every } \mathbf{p} \geq \mathbf{0}_I \text{ and } \mathbf{x} \geq \mathbf{0}_J\}.$

Thus the value-added function V may be used in order to describe completely a firm's technology, provided that the firm's underlying production possibilities set $\boldsymbol{T}$ satisfies the regularity-conditions given by 4.1.

Under certain circumstances, we may use the value-added function V in order to define a *real* value-added function which has the usual neoclassical production function properties. Suppose the prices of produced goods $\mathbf{p}$ vary in strict proportion, i.e., $\mathbf{p} = p_0\boldsymbol{\alpha}$ where $\boldsymbol{\alpha} \gg \mathbf{0}_I$ is a

[31]See Diewert (1973a) for a proof. Condition (i) of 4.4 is equivalent to condition (i) of 2.21 in Diewert (1973) once the domain of definition $\Pi(p;v)$ is extended by continuity from $\mathbf{p} \gg \mathbf{0}_I$ to $\mathbf{p} \geq \mathbf{0}_I$.

[32]See Diewert (1973a) for a proof under slightly different but equivalent regularity conditions on the variable profit function.

fixed vector of constants and p_0 is a scalar which may vary from period to period. Then a *real value-added* function f_α may be defined by

(4.6) $f_\alpha(x) \equiv V(p_0\boldsymbol{\alpha};\mathbf{x})/p_0, \quad \boldsymbol{\alpha} \gg \mathbf{0}_I, \quad \mathbf{x} \geq \mathbf{0}_J, \quad p_0 > 0.$

Note that $f_\alpha(x)$ is an observable quantity; it is equal to (nominal) value-added V divided by the price index p_0. Note also that if V satisfies Conditions II, then $V(p_0\boldsymbol{\alpha};\mathbf{x})/p_0 = V(\boldsymbol{\alpha};\mathbf{x})$, using (ii) of 4.4.

(4.7) *Theorem* [Khang (1971), Bruno (Chapter III.1)]. If the production possibilities set $\boldsymbol{T}$ satisfies Conditions I and if $V(\boldsymbol{\alpha};\mathbf{x}^*) > 0$ for some $\mathbf{x}^* \geq \mathbf{0}_J$ where $\boldsymbol{\alpha} \gg \mathbf{0}_I$ (i.e., nominal value-added is positive for at least one vector of fixed inputs), then the real value-added function f_α defined by equation 4.6 is: (i) linear homogeneous in $\mathbf{x}$, (ii) non-decreasing in $\mathbf{x}$, (iii) a continuous concave function of $\mathbf{x}$, and (iv) positive if $\mathbf{x} \gg \mathbf{0}_J$.

Proof: Properties (i), (ii), and (iii) follow directly from properties (iv), (v), and (vi) of the (nominal) value-added function V. Proof of (iv). Let $\mathbf{x} \gg \mathbf{0}_J$. Then there exists $\lambda > 0$ such that $\lambda\mathbf{x} \geq \mathbf{x}^*$. Therefore $0 < f_\alpha(\mathbf{x}^*) \equiv V(\boldsymbol{\alpha};\mathbf{x}^*) \leq V(\boldsymbol{\alpha};\lambda\mathbf{x})$ [using (v) of 4.4] $= \lambda V(\boldsymbol{\alpha};\mathbf{x})$ [using (iv) of 4.4] $= \lambda f_\alpha(\mathbf{x})$. Since $\lambda > 0$, $f_\alpha(\mathbf{x}) > 0$ also. Q.E.D.

Thus if prices of outputs and intermediate inputs vary in strict proportion over time, the deflated nominal value-added function $V(p_0\boldsymbol{\alpha};\mathbf{x})/p_0 \equiv f_\alpha(\mathbf{x})$ is a perfectly well behaved neoclassical production function and the substitution of deflated value-added for real output can be justified from the viewpoint of empirical production function studies.

If prices of outputs and intermediate inputs do not vary in strict proportion, then the parameters of technology can still be estimated, provided that we have information on value-added V, on produced goods prices $\mathbf{p}$ and on primary inputs $\mathbf{x}$ – simply assume a functional form for $V(\mathbf{p};\mathbf{x})$ and use regression techniques in order to estimate the parameters of the functional form.[33]

In the following section, we consider the problem of minimizing the cost of producing a given amount of nominal value-added, and we develop a duality between the value-added function V and the unit value-added cost function. This last function may be used in order to obtain systems of demand functions for inputs which are consistent with cost minimization.

[33]See Diewert (1973a and 1974a) for some functional forms for V (or equivalently for Π) which are linear in the unknown parameters.

5. The Duality between Value-Added "Production" and Cost Functions

Throughout this section, we assume that the value-added function $V(\mathbf{p};\mathbf{x})$ satisfies Conditions II given by 4.4 in the previous section.

In general, value-added will not be positive for all output price vectors $\mathbf{p} \geq \mathbf{0}_I$; i.e., if the prices of intermediate inputs are high relative to the prices of goods that the firm produces, then it may not pay the firm to produce anything at all and value-added will be zero. These considerations lead us to define the following set of prices where value-added is zero:

(5.1) $\boldsymbol{P} \equiv \{\mathbf{p}: V(\mathbf{p};\mathbf{1}) = 0; \mathbf{p} \geq \mathbf{0}_I\}$,

where $\mathbf{1}$ is a J-dimensional vector of primary inputs which has each component equal to unity.

(5.2) *Lemma.* If V satisfies Conditions II, then $\mathbf{P}$ defined by equation 5.1 is a non-empty, closed convex cone.

Proof: Since $V(\mathbf{p};\mathbf{x})$ is continuous and linear homogeneous in $\mathbf{p}$, we have $V(\mathbf{0}_I;\mathbf{1}) = 0$ and thus $\mathbf{0}_I \in \boldsymbol{P}$. Let $\mathbf{p}',\mathbf{p}'' \in \boldsymbol{P}$ and let $0 \leq \lambda \leq 1$. Since V is non-negative, we have $0 \leq V(\lambda\mathbf{p}' + (1-\lambda)\mathbf{p}'';\mathbf{1}) \leq \lambda V(\mathbf{p}';\mathbf{1}) + (1-\lambda)V(\mathbf{p}'';\mathbf{1})$ (using the convexity of V in $\mathbf{p}$) $= \lambda\cdot 0 + (1-\lambda)\cdot 0 = 0$ and thus $\boldsymbol{P}$ is a convex set. That $\boldsymbol{P}$ is a cone follows from the linear homogeneity of V in $\mathbf{p}$ and the closedness of $\boldsymbol{P}$ follows from the continuity of V in $\mathbf{p}$. Q.E.D.

We note that the set $\boldsymbol{P}$ could reduce to the single point $\{\mathbf{0}_I\}$.

Given that a firm is producing a certain amount of (nominal) value added, it is natural to suppose that under certain circumstances the firm will attempt to choose a combination of primary inputs which will produce the value-added at minimum cost. Thus we define the *unit value-added cost function*:

(5.3) $$c(\mathbf{p};\mathbf{w}) \equiv \min_{\mathbf{x}}\{\mathbf{w}^T\mathbf{x}: V(\mathbf{p};\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} \quad \text{if} \quad \mathbf{p} \geq \mathbf{0}_I, \mathbf{p} \notin \boldsymbol{P},$$
$$\equiv +\infty \quad \text{if} \quad \mathbf{p} \in \boldsymbol{P},$$

where $\mathbf{w} \gg \mathbf{0}_J$ is a positive vector of given primary input prices, V is the value-added function and $\boldsymbol{P}$ is the set of prices defined by equation 5.1.

(5.4) *Lemma.* The unit value-added cost function is well-defined as a minimum, provided that $\mathbf{p} \notin \boldsymbol{P}$.

Proof: Let $\mathbf{p}^* \geq \mathbf{0}_I$, $\mathbf{w}^* \gg \mathbf{0}_J$ and $\mathbf{p}^* \notin \boldsymbol{P}$. Then $V(\mathbf{p}^*;\mathbf{1}) > 0$ and using the linear homogeneity of $V(\mathbf{p};\mathbf{x})$ in $\mathbf{x}$, we have $V(\mathbf{p}^*;\lambda^*\mathbf{1}) = 1$ where $\lambda^* \equiv V(\mathbf{p}^*;\mathbf{1})$. Thus the set $\{\mathbf{x}: V(\mathbf{p}^*;\mathbf{x}) \geq 1,\ \mathbf{x} \geq \mathbf{0}\}$ is not empty and is closed using the continuity of $V(\mathbf{p};\mathbf{x})$ in $\mathbf{x}$. We have $c(\mathbf{p}^*;\mathbf{w}^*) \equiv \min_{\mathbf{x}}\{\mathbf{w}^{*T}\mathbf{x}: V(\mathbf{p}^*;\mathbf{x}) \geq 1,\ \mathbf{x} \geq \mathbf{0},\ \mathbf{w}^{*T}\mathbf{x} \leq \mathbf{w}^{*T}\lambda^*\mathbf{1}\}$, and the minimum of a linear function over a closed, bounded non-empty set is attained.
Q.E.D.

If the prices of produced goods are such that the firm cannot produce a positive amount of value-added (i.e., $\mathbf{p} \in \boldsymbol{P}$), then it is natural to define the cost of producing one unit of value-added to be plus infinity. However, the above lemma shows that as long as $\mathbf{p} \notin \boldsymbol{P}$ and $\mathbf{w} \gg \mathbf{0}$, the unit value-added cost function is well-defined as a minimum. Note that since $V(\mathbf{p};\mathbf{x})$ is linearly homogeneous in $\mathbf{x}$, in order to find the minimum cost of producing $\bar{V}$ units of nominal value-added, simply multiply $c(\mathbf{p};\mathbf{w})$ by $\bar{V}$.

(5.5) *Theorem.* If the value-added function V satisfies Conditions II given by 4.4, then the unit value-added cost function defined by equation 5.3 satisfies the following conditions:

(5.6) *Conditions III* on the Unit Value-Added Cost Function c:

(i) $c(\mathbf{p};\mathbf{w})$ is a *positive* extended real valued function defined for $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \gg \mathbf{0}_J$, and is infinite for $\mathbf{p} \in \boldsymbol{P}$ where $\boldsymbol{P}$ is a closed convex cone of prices.

(ii) c is *continuous from above* in $(\mathbf{p},\mathbf{w})$.

(iii) c is *quasi-concave* in $(\mathbf{p},\mathbf{w})$.

(iv) c is *homogeneous of degree* -1 in $\mathbf{p}$ for fixed $\mathbf{w}$, i.e., for $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \gg \mathbf{0}_J$ and $\lambda > 0$, we have $c(\lambda\mathbf{p};\mathbf{w}) = \lambda^{-1}c(\mathbf{p};\mathbf{w})$.

(v) c is *homogeneous of degree* 1 in $\mathbf{w}$ for fixed $\mathbf{p}$, i.e., for $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \gg \mathbf{0}_J$ and $\lambda > 0$, we have $c(\lambda\mathbf{p};\mathbf{w}) = \lambda^{-1}c(\mathbf{p};\mathbf{w})$.

(vi) c is *non-decreasing* in $\mathbf{w}$ for fixed $\mathbf{p}$.

A proof of Theorem 5.5 is given in Section 7.

Let us relate the properties of the unit value-added cost function to something more familiar. Let $\mathbf{p} = p_0\boldsymbol{\alpha}$ where $p_0 > 0$ and $\boldsymbol{\alpha} \geq \mathbf{0}_I$ is such that $\boldsymbol{\alpha} \notin \boldsymbol{P}$. Then given primary input prices $\mathbf{w} \gg \mathbf{0}_J$, the minimum cost of producing one unit of *real* value-added may be defined as $c(\boldsymbol{\alpha};\mathbf{w}) \equiv \min_{\mathbf{x}}\{\mathbf{w}^T\mathbf{x}: V(p_0\boldsymbol{\alpha};\mathbf{x}) \geq p_0 \cdot 1,\ \mathbf{x} \geq \mathbf{0}_J\}$ where c is the unit value-added cost function. It can be seen that $c(\boldsymbol{\alpha};\mathbf{w})$ regarded as a function of $\mathbf{w}$ alone is a positive, homogeneous of degree one, non-decreasing, concave[34] and continuous[35] function of $\mathbf{w} \gg \mathbf{0}_J$; i.e., it has the properties of a unit cost function which is dual to a constant returns to scale production function. [In fact, $c(\boldsymbol{\alpha};\mathbf{w})$ may be interpreted as the unit cost function which is dual to the real value-added production function $f_{\alpha}(\mathbf{x})$ defined by equation 4.6.]

It turns out that the unit value-added cost function c may be used in order to generate the value-added function of the technology. Let us suppose that c satisfies Conditions III. Then we may (uniquely) extend the domain of definition of c from $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \gg \mathbf{0}_J$ to $(\mathbf{p},\mathbf{w}) \geq \mathbf{0}_{I+J}$ by continuity. Having done this, we may define the function V^* for $\mathbf{p} \geq \mathbf{0}_I$ and $\mathbf{x}^* \gg \mathbf{0}_J$ as follows:

(5.7)
$$\begin{aligned} V^*(\mathbf{p};\mathbf{x}^*) &\equiv \max_{\lambda}\{\lambda : c(\mathbf{p};\mathbf{w})\lambda \leq \mathbf{w}^T\mathbf{x}^* \text{ for every } \mathbf{w} \geq \mathbf{0} \text{ such that } \\ &\qquad \mathbf{w}^T\mathbf{x}^* = 1\} \\ &= \max_{\lambda}\left\{\lambda : \lambda \leq \frac{\mathbf{w}^T\mathbf{x}^*}{c(\mathbf{p};\mathbf{w})} \text{ for every } \mathbf{w} \geq \mathbf{0} \text{ such that } \mathbf{w}^T\mathbf{x}^* = 1\right\} \\ &= \min_{\mathbf{w}}\left\{\frac{1}{c(\mathbf{p};\mathbf{w})} : \mathbf{w} \geq \mathbf{0}_J; \mathbf{w}^T\mathbf{x}^* = 1\right\} \\ &= \min_{\mathbf{w}}\left\{\frac{1}{c(\mathbf{p};\mathbf{w})} : \mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x}^* \leq 1\right\}. \end{aligned}$$

In developing the string of equalities in equations 5.7, we have used the facts that c is non-negative, homogeneous of degree one in $\mathbf{w}$ and non-decreasing in $\mathbf{w}$.

(5.8) *Theorem.* If c satisfies Conditions III given by 5.6, then $V^*(\mathbf{p};\mathbf{x})$ defined by equations 5.7 satisfies Conditions II given by 4.4 over the domain $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{x} \gg \mathbf{0}_J$. Moreover, if c was generated by means of Definition 5.3 for some V which satisfied Conditions II, then $V^* = V$.

We note that the domain of definition of the function V^* can be (uniquely) extended to $(\mathbf{p};\mathbf{x}) \geq \mathbf{0}_{I+J}$ by continuity. A proof of Theorem 5.8 may be found in Section 7.

[34]A function which is positive, quasi-concave and (positively) homogeneous of degree one over a set is concave over that set. See Berge (1963, p. 208).

[35]A function which is concave over an open convex set is continuous over that set. See Rockafellar (1970, p. 82).

Theorems 5.5 and 5.8 establish a duality between a value-added function satisfying Conditions II and a unit value-added cost function satisfying Conditions III which is similar to the Shephard (1953 and 1970) duality theorem between cost and production functions.

The usefulness of the unit value-added cost function may be found in the following result:[36]

(5.9) *Lemma* [Shephard (1953)]. Suppose that c satisfies Conditions III, is finite at the prices $\mathbf{p}^* \geq \mathbf{0}_I$, $\mathbf{w}^* \gg \mathbf{0}_J$, and is once continuously differentiable with respect to the input prices $\mathbf{w}$ at $\mathbf{w} = \mathbf{w}^*$. Then the cost-minimizing amount of primary input j, $x_j(V;\mathbf{p}^*;\mathbf{w}^*)$, given that value-added V is to be produced, is given by $x_j(V;\mathbf{p}^*;\mathbf{w}^*) \equiv V \cdot \partial c(\mathbf{p}^*;\mathbf{w}^*)/\partial w_j$ for $j = 1,2,\ldots,J$.

Thus in order to obtain a system of primary input demand functions which is consistent with the value-added being maximized while primary input cost is minimized, we need only assume a functional form for c which is consistent with Conditions III and in addition is differentiable with respect to primary input prices, and then apply Lemma 5.9.

Finally, we note that Theorems 5.5 and 5.8 reduce to a variant of the Shephard Duality Theorem[37] between unit cost and constant returns to scale production functions, if we assume that only one output is produced (so that $I = 1$). In this case, the vector $\mathbf{p} \geq \mathbf{0}_I$ reduces to a scalar and if we set $p = 1$, the value added function $V(1;\mathbf{x}) \equiv f(\mathbf{x})$ becomes a linear homogeneous production function which satisfies the same regularity conditions as the unit cost function $c(1;\mathbf{w})$.

6. Concluding Remarks

In Section 2 above, we saw that the use of aggregate commodities in consumer demand studies could be justified without making any restrictive assumptions on the functional form of the micro utility function if the prices of the micro commodities within a group varied in strict

[36]See Diewert (1971, p. 495) for a proof (originally due to L. McKenzie) which carries over into the present context and for a more detailed list of historical references.

[37]See (i) Shephard (1953) for a proof which assumed differentiability of the production and unit cost functions, (ii) Samuelson (1953–54, p. 15) for a statement of the theorem, and (iii) Shephard (1970) for a proof which does not assume differentiability. Duality theorems between non-homogeneous production functions and total cost functions have been proven by Shephard (1953 and 1970), Diewert (1971), Hanoch (Chapter I.2), and McFadden (Chapter I.1).

proportion. This condition is not without empirical relevance, since in practice, the prices of many micro commodities do vary proportionately, at least approximately.[38] Thus the work of national income accountants would be of maximum benefit to the applied econometrician concerned with estimating systems of consumer demand functions if microeconomic consumer budget study data were aggregated in a manner which conformed as closely as possible with the hypotheses of Hicks' Aggregation Theorem.[39]

In Section 4 above, we saw that the replacement of real output by deflated value-added in production function studies could be justified without making any restrictive assumptions on the functional form of the micro production (or transformation) function, provided that prices of outputs and intermediate inputs varied in strict proportion. However, it is extremely unlikely that this last condition has been satisfied in any western industrial economy during the past twenty-five years. The problem is with the intermediate input, *energy*. The price of energy has generally been constant or growing very slowly during the pre-1973 postwar period, and thus the price of energy relative to the price of produced goods in general has fallen.[40] Also the industrial consumption of energy has risen faster than the rate of growth of output in general,[41] and thus one cannot appeal to Leontief's Aggregation Theorem (1936, p. 55) in order to justify the substitution of deflated value-added for real gross output.

Recall the separability approach of Corden (1969), Sims (1969) and Arrow (1974) where the aggregate production function $f(K,L,M)$ was assumed to be of the form $f(K,L,M) \equiv g(h(K,L),M)$ where $h(K,L)$ was identified as real value-added, where we now regard M as a vector of intermediate inputs including energy, K is a vector of capital service inputs and L is a vector of labour inputs. Berndt and Christensen (1973b) have shown that the assumption of separability leads to severe restrictions on partial elasticities of substitution[42] between pairs of inputs. If

[38]The prices of two micro commodities can vary in proportion over time for at least two reasons: the goods may be close substitutes in consumption (e.g., butter and margarine) or the goods may be close substitutes in production (e.g., cars and trucks).

[39]Thus if a major change in the relative prices of micro commodities within an aggregate occurred, then the aggregate should be decomposed into at least two subaggregates. For example, the introduction of medicare would radically change the relative prices of the micro commodities in a household services aggregate.

[40]See the price data on energy, other intermediate inputs and output for U.S. manufacturing tabled in Berndt and Wood (1975) for the period 1950 to 1970.

[41]See Berndt and Wood (1975).

[42]See Allen (1938, p. 503–509) for a definition.

the above separability assumption $f(K,L,M) = g(h(K,L),M)$ is satisfied, then it turns out that the elasticity of substitution between energy and *any* capital or labour input must be the same number. This condition seems highly implausible – we would expect energy and certain types of capital services to be complements (i.e., have negative partial elasticities of substitution), energy and certain types of maintenance workers to be complements, while energy and unskilled labour would be expected to be substitutes (i.e., have positive partial elasticities of substitution). Thus the separability approach to justifying the use of "real" value-added in place of output in production function studies would also seem to be unacceptable on *a priori* empirical grounds.

Our conclusion is that most postwar production function studies that use deflated value-added as a measure of real output are probably somewhat biased due to their inadequate treatment of energy inputs. One is also led to wonder about how much of the "unexplained residual" in growth studies (sometimes called "technical progress") is due to the unjustified use of a real value-added framework.[43]

Finally, from Theorem 3.1, it can be seen that if the direct utility function is continuous, then the conditional indirect utility function $U_{\alpha}(y_0,\mathbf{y})$ is continuous with respect to the vector of price proportionality factors, $\boldsymbol{\alpha} \gg \mathbf{0}_M$. Similarly, from Definition 4.6, $f_{\beta}(\mathbf{x}) \equiv V(p_0\boldsymbol{\beta},\mathbf{x})/p_0$, and Condition 4.4(iii) on the nominal value-added function V, it can be seen that the real value-added function $f_{\beta}(\mathbf{x})$ is continuous with respect to the vector of price proportionality factors $\boldsymbol{\beta} \gg \mathbf{0}_I$.

The above continuity properties imply that approximate versions of Hicks' Aggregation Theorem will hold in both the consumer and producer contexts. Thus if a group of consumer prices vary over time *approximately* according to the vector of proportionality factors $\boldsymbol{\alpha}$, then the consumer's preferences can *approximately* be represented by the aggregated utility function $U_{\alpha}(y_0,\mathbf{y})$. Similarly, if a group of producer prices vary over time *approximately* according to the vector of proportionality factors $\boldsymbol{\beta}$, then the producer's technology can *approximately* be represented by the real value-added function $f_{\beta}(\mathbf{x})$.[44] Just how much variation in $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ we can allow before the approximation becomes poor

[43]Two notable studies which use a gross output formulation rather than value-added are Parks (1971) and Berndt and Wood (1975).

[44]In Diewert (1974c), we show that an aggregate elasticity of substitution between an aggregated "macro" good and a "micro" good can be written as a weighted sum of micro elasticities of substitution under the hypotheses of Hicks' Aggregation Theorem and we indicate how this aggregate elasticity will change if prices do not vary proportionately.

must remain an empirical matter. Much would depend on the variation in prices and quantities of the goods which are not aggregated relative to the variability in the prices of the goods within the aggregate.

7. Proofs of Theorems

Proof of Theorem 2.4

(i) The result follows directly from the Upper Semicontinuity Maximum Theorem 1.8.

(ii) Let $\alpha \gg 0_M$, $w \gg 0_N$, $p_0 > 0$, $Y \geq 0$. We have $f(x^*,y^*) = \max_{x,y}\{f(x,y): p_0\alpha^T x + w^T y \leq Y, \quad x \geq 0_M, \quad y \geq 0_N\} = \max_{x,y,y_0}\{f(x,y): p_0 y_0 + w^T y \leq Y, \quad \alpha^T x \leq y_0, \quad x \geq 0_M, \quad y \geq 0_N, \quad y_0 \geq 0\} = \max_{y,y_0}\{U_\alpha(y_0,y): p_0 y_0 + w^T y \leq Y, y \geq 0_N, y_0 \geq 0\} = U_\alpha(\alpha^T x^*, y^*)$.

(iii–a) Let $\alpha \gg 0_M$, $(y_0^*,y^*) \geq 0_{N+1}$ and let $\epsilon > 0$. $U_\alpha(y_0^*,y^*) \equiv \max_x\{f(x,y^*): x \geq 0_M, \alpha^T x \leq y_0^*\} = f(x^*,y^*)$, say. For every $\delta > 0$, we can find $(x_\delta,y_\delta) \geq 0_{M+N}$ such that $f(x_\delta,y_\delta) > f(x^*,y^*)$ and $(x_\delta - x^*)^T(x_\delta - x^*) + (y_\delta - y^*)^T(y_\delta - y^*) \leq \delta^2$. Define $y_0(\delta) \equiv \alpha^T x_\delta$. By the definition of U_α, we have $U_\alpha(y_0(\delta),y_\delta) \geq f(x_\delta,y_\delta) > f(x^*,y^*) = U_\alpha(y_0^*,y^*)$ and $(y_0(\delta) - y_0^*)^2 + (y_\delta - y^*)^T(y_\delta - y^*) = (x_\delta - x^*)^T\alpha\alpha^T(x_\delta - x^*) + (y_\delta - y^*)^T(y_\delta - y^*) \leq \epsilon^2$ for δ small enough.

(iii–b) Continuity of U_α follows directly from the Maximum Theorem 1.9.

(iii–c) Let $y_0'' \geq y_0' \geq 0$, $y'' \geq y' \geq 0_N$ and $\alpha \gg 0_M$. Then $U_\alpha(y_0',y') \equiv \max_x\{f(x,y'): \alpha^T x \leq y_0'; \quad x \geq 0_M\} = f(x',y') \leq f(x',y'')$ [since $y' \leq y''$] $\leq \max_x\{f(x,y''): \alpha^T x \leq y_0'; \; x \geq 0_M\}$ [since $\alpha^T x' \leq y_0'$] $\leq \max_x\{f(x,y''): \alpha^T x \leq y_0''; \; x \geq 0_M$ [since $y_0'' \geq y_0'$] $= U_\alpha(y_0'',y'')$.

(iii–d) Let $y_0' \geq 0$, $y_0'' \geq 0$, $y' \geq 0_N$, $y'' \geq 0_N$, and $\alpha \gg 0_M$. Suppose that $U_\alpha(y_0',y') \equiv \max_x\{f(x,y'): \alpha^T x \leq y_0'; \; x \geq 0_M\} = f(x',y') \geq k$ and $U_\alpha(y_0'',y'') = f(x'',y'') \geq k$ where k is a scalar. Let $0 \leq \lambda \leq 1$. Then $U_\alpha(\lambda y_0' + (1-\lambda)y_0'', \lambda y' + (1-\lambda)y'') \equiv \max_x\{f(x, \lambda y' + (1-\lambda)y''): \alpha^T x \leq \lambda y_0' + (1-\lambda)y_0'', x \geq 0_M\} \geq f(\lambda x' + (1-\lambda)x'', \quad \lambda y' + (1-\lambda)y'')$ [since $\lambda\alpha^T x' + (1-\lambda)\alpha^T x'' \leq \lambda y_0' + (1-\lambda)y_0''] \geq k$ (using the quasi-concavity of f].

(iii–e) The proof is similar to (iii–d) above, upon noting that $(y_0',y') \neq (y_0'',y'')$ implies that $(x',y') \neq (x'',y'')$.

Proof of Theorem 3.1

(i) The result follows directly from the Upper Semicontinuity Maximum Theorem 1.8.

(ii) Let $y_0 \geq 0$, $\mathbf{y} \geq \mathbf{0}_N$, $\boldsymbol{\alpha}'' \geq \boldsymbol{\alpha}' \gg \mathbf{0}_M$. Then $U_{\alpha''}(y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \geq \mathbf{0}_M, \boldsymbol{\alpha}''^T\mathbf{x} \leq y_0\} \leq \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \geq \mathbf{0}_M, \boldsymbol{\alpha}'^T\mathbf{x} \leq y_0\}$ [since $\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}''$ and thus the set of feasible x's has not decreased] $= U_{\alpha'}(y_0,\mathbf{y})$.

(iii) Let $y_0 \geq 0$, $\mathbf{y} \geq \mathbf{0}_N$, $\boldsymbol{\alpha}'' \gg \mathbf{0}_M$, $\boldsymbol{\alpha}' \gg \mathbf{0}_M$, $0 \leq \lambda \leq 1$, and define the sets $\mathbf{H}' \equiv \{\mathbf{x}:\boldsymbol{\alpha}'^T\mathbf{x} \leq y_0;\ \ \mathbf{x} \geq \mathbf{0}_M\}$, $\boldsymbol{H}'' \equiv \{\mathbf{x}:\boldsymbol{\alpha}''^T\mathbf{x} \leq y_0;\ \ \mathbf{x} \geq \mathbf{0}_M\}$ and $\boldsymbol{H}^\lambda \equiv \{\mathbf{x}:(\lambda\boldsymbol{\alpha}' + (1-\lambda)\boldsymbol{\alpha}'')^T\mathbf{x} \leq y_0;\ \mathbf{x} \geq \mathbf{0}_M\}$. We first show that $\boldsymbol{H}^\lambda$ is a subset of the union of $\boldsymbol{H}'$ and $\boldsymbol{H}''$. Suppose $\mathbf{x}^* \in \boldsymbol{H}^\lambda$, but $\mathbf{x}^* \notin \boldsymbol{H}'$ and $\mathbf{x}^* \notin \boldsymbol{H}''$. Thus $(\lambda\boldsymbol{\alpha}' + (1-\lambda)\boldsymbol{\alpha}'')^T\mathbf{x}^* \leq y_0$ and $\mathbf{x}^* \geq \mathbf{0}_M$ since $\mathbf{x}^* \in \boldsymbol{H}^\lambda$. If $\mathbf{x}^* \notin \boldsymbol{H}'$, $\mathbf{x}^* \notin \boldsymbol{H}''$, then since $\mathbf{x}^* \geq \mathbf{0}_M$, we must have $\boldsymbol{\alpha}'^T\mathbf{x}^* > y_0$, $\boldsymbol{\alpha}''^T\mathbf{x}^* > y_0$. Thus $(\lambda\boldsymbol{\alpha}' + (1-\lambda)\boldsymbol{\alpha}'')^T\mathbf{x}^* > y_0$ which contradicts the supposition $\mathbf{x}^* \in \boldsymbol{H}^\lambda$. Thus $\boldsymbol{H}^\lambda \subset \boldsymbol{H}' \cup \boldsymbol{H}''$. Now let $U_{\alpha'}(y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \in \boldsymbol{H}'\} \leq k$, and $U_{\alpha''}(y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \in \boldsymbol{H}''\} \leq k$ where k is a scalar. Then $U_{\lambda\alpha'+(1-\lambda)\alpha''}(y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\mathbf{x} \in \boldsymbol{H}^\lambda\} \leq \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y});\mathbf{x} \in \boldsymbol{H}' \cup \boldsymbol{H}''\}$ [since $\boldsymbol{H}^\lambda \subset \boldsymbol{H}' \cup \boldsymbol{H}''] \leq k$.

(iv) Let $\mathbf{y} \geq \mathbf{0}_N$, $\boldsymbol{\alpha} \gg \mathbf{0}_M$, $\lambda > 0$, $y_0 \geq 0$. Then $U_{\lambda\alpha}(\lambda y_0,\mathbf{y}) \equiv \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\lambda\boldsymbol{\alpha}^T\mathbf{x} \leq \lambda y_0;\ \ \mathbf{x} \geq \mathbf{0}_M\} = \max_{\mathbf{x}}\{f(\mathbf{x},\mathbf{y}):\boldsymbol{\alpha}^T\mathbf{x} \leq y_0;\ \ \mathbf{x} \geq \mathbf{0}_M\}$ [since $\lambda > 0] = U_\alpha(y_0,\mathbf{y})$.

(v) The result follows directly from the Maximum Theorem 1.9.

Proof of Theorem 3.5

Let f be positively linearly homogéneous, $\mathbf{v} \gg \mathbf{0}_M$, $\mathbf{y} \geq \mathbf{0}_N$ and $\lambda > 0$. Then

$$\begin{aligned} g(\lambda^{-1}\mathbf{v},\lambda\mathbf{y}) &\equiv \max_{\mathbf{x}}\{f(\mathbf{x},\lambda\mathbf{y}):\lambda^{-1}\mathbf{v}^T\mathbf{x} \leq 1, \mathbf{x} \geq \mathbf{0}_M\} \\ &= \max_{\lambda\mathbf{z}}\{f(\lambda\mathbf{z},\lambda\mathbf{y}):\lambda^{-1}\mathbf{v}^T\lambda\mathbf{z} \leq 1, \lambda\mathbf{z} \geq \mathbf{0}_M\} \\ &= \max_{\mathbf{z}}\{\lambda f(\mathbf{z},\mathbf{y}):\mathbf{v}^T\mathbf{z} \leq 1, \mathbf{z} \geq \mathbf{0}_M\} \\ &= \lambda g(\mathbf{v},\mathbf{y}). \end{aligned}$$

Now let g have the stated property and let $\mathbf{x} \gg \mathbf{0}_M$, $\mathbf{y} \geq \mathbf{0}_N$, $\lambda > 0$. Then

$$\begin{aligned} f(\lambda\mathbf{x},\lambda\mathbf{y}) &\equiv \min_{\mathbf{v}}\{g(\mathbf{v},\lambda\mathbf{y}):\mathbf{v}^T\lambda\mathbf{x} \leq 1, \mathbf{v} \geq \mathbf{0}_M\} \\ &= \min_{\lambda^{-1}\mathbf{z}}\{g(\lambda^{-1}\mathbf{z},\lambda\mathbf{y}):\lambda^{-1}\mathbf{z}^T\lambda\mathbf{x} \leq 1, \lambda^{-1}\mathbf{z} \geq \mathbf{0}_M\} \\ &= \min_{\mathbf{z}}\{\lambda g(\mathbf{z},\mathbf{y}):\mathbf{z}^T\mathbf{x} \leq 1, \mathbf{z} \geq \mathbf{0}_M\} \\ &= \lambda f(\mathbf{x},\mathbf{y}). \end{aligned}$$

Proof of Theorem 3.18

Part 1. We show that concavity of $f(\mathbf{x},\mathbf{y})$ in $\mathbf{x},\mathbf{y}$ implies that $C(u;\mathbf{p},\mathbf{y})$ is convex $u,\mathbf{y}$ for every $\mathbf{p} \gg \mathbf{0}_M$. Let $\mathbf{p} \gg \mathbf{0}_M$, $\mathbf{y}^1 \geq \mathbf{0}_N$, $\mathbf{y}^2 \geq \mathbf{0}_N$, $0 \leq \lambda \leq 1$, and define $\mathbf{x}^i$, $i = 1,2$, by

$$\text{(i)} \quad C(u^i;\mathbf{p},\mathbf{y}^i) \equiv \min_{\mathbf{x}}\{\mathbf{p}^T\mathbf{x}: f(\mathbf{x},\mathbf{y}^i) \geq u^i, \mathbf{x} \geq \mathbf{0}_M\} \equiv \mathbf{p}^T\mathbf{x}^i.$$

Then

$$\begin{aligned}
\text{(i}'\text{)} \quad & C(\lambda u^1 + (1-\lambda)u^2;\mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \\
&\equiv \min_{\mathbf{x}}\{\mathbf{p}^T\mathbf{x}: f(\mathbf{x},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \geq \lambda u^1 + (1-\lambda)u^2\} \\
&\leq \mathbf{p}^T(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2) \\
&= \lambda C(u^1;\mathbf{p},\mathbf{y}^1) + (1-\lambda)C(u^2;\mathbf{p},\mathbf{y}^2),
\end{aligned}$$

since by concavity of f, $f(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2, \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \geq \lambda f(\mathbf{x}^1,\mathbf{y}^1) + (1-\lambda)f(\mathbf{x}^2,\mathbf{y}^2) \geq \lambda u^1 + (1-\lambda)u^2$ [using (i)], and thus $\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ is feasible for the minimization problem (i). Throughout this proof, u^i's are assumed to belong to the range of the direct utility function.

Part 2. We show that convexity of $C(u;\mathbf{p},\mathbf{y})$ in $u,\mathbf{y}$ for every $\mathbf{p} \gg \mathbf{0}_M$ implies that $g(\mathbf{p}/Y,\mathbf{y}) \equiv \max_u\{u: C(u;\mathbf{p}/Y,\mathbf{y}) \leq 1\}$ is concave in $Y > 0$, $\mathbf{y} \geq \mathbf{0}_N$ for every $\mathbf{p} \gg \mathbf{0}_M$. Let $\mathbf{p} \gg \mathbf{0}_M$, $\mathbf{y}^1 \geq \mathbf{0}_N$, $\mathbf{y}^2 \geq \mathbf{0}_N$, $Y^1 > 0$, $Y^2 > 0$, $0 \leq \lambda \leq 1$, and define u^i, $i = 1,2$, by

$$\text{(ii)} \quad g(\mathbf{p}/Y^i,\mathbf{y}^i) \equiv \max_u\{u: C(u;\mathbf{p}/Y^i,\mathbf{y}^i) \leq 1\} \equiv u^i.$$

Then

$$\begin{aligned}
\text{(ii}'\text{)} \quad & g(\mathbf{p}/(\lambda Y^1 + (1-\lambda)Y^2), \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \\
&\equiv \max_u\{u: C(u;\mathbf{p}/(\lambda Y^1 + (1-\lambda)Y^2), \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \leq 1\} \\
&= \max_u\{u: C(u;\mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \leq \lambda Y^1 + (1-\lambda)Y^2\} \\
&\geq \lambda u^1 + (1-\lambda)u^2 \\
&= \lambda g(\mathbf{p}/Y^1,\mathbf{y}^1) + (1-\lambda)g(\mathbf{p}/Y^2,\mathbf{y}^2),
\end{aligned}$$

since by the convexity of C, $C(\lambda u^1 + (1-\lambda)u^2;\mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \leq \lambda C(u^1;\mathbf{p},\mathbf{y}^1) + (1-\lambda)C(u^2;\mathbf{p},\mathbf{y}^2) \leq \lambda Y^1 + (1-\lambda)Y^2$ [using (ii)], and thus $\lambda u^1 + (1-\lambda)u^2$ is feasible for the maximization problem (ii′).

Part 3. We show that $g(\mathbf{p}/Y,y)$ concave in $Y>0$, $\mathbf{y}\geq \mathbf{0}_N$ for every $\mathbf{p}\gg \mathbf{0}_M$ implies that the restricted cost function $C(u;\mathbf{p},\mathbf{y}) \equiv \min_Y\{g(\mathbf{p}/Y,\mathbf{y})\geq u,\ Y>0\}$ is convex in u, $\mathbf{y}\geq \mathbf{0}_N$ for every $\mathbf{p}\gg \mathbf{0}_M$. Let $\mathbf{p}\gg \mathbf{0}_M$, $\mathbf{y}^1\geq \mathbf{0}_N$, $\mathbf{y}^2\geq \mathbf{0}_N$, $0\leq\lambda\leq 1$, and define Y^i, $i=1,2$, by

(iii) $C(u^i;\mathbf{p},\mathbf{y}^i) \equiv \min_Y\{Y: g(\mathbf{p}/Y,\mathbf{y}^i)\geq u^i\} \equiv Y^i.$

Then

(iii′)
$$\begin{aligned} &C(\lambda u^1 + (1-\lambda)u^2;\ \mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2)\\ &\equiv \min_Y\{Y: g(\mathbf{p}/Y,\ \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2)\geq \lambda u^1 + (1-\lambda)u^2\}\\ &\leq \lambda Y^1 + (1-\lambda)Y^2\\ &= \lambda C(u^1;\mathbf{p},\mathbf{y}^1) + (1-\lambda)C(u^2;\mathbf{p},\mathbf{y}^2), \end{aligned}$$

since by the concavity property of g, $g(\mathbf{p}/(\lambda Y^1 + (1-\lambda)Y^2),\ \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \geq \lambda g(\mathbf{p}/Y^1,\mathbf{y}^1) + (1-\lambda)g(\mathbf{p}/Y^2,\mathbf{y}^2) \geq \lambda u^1 + (1-\lambda)u^2$ [using (iii)], and thus $\lambda Y^1 + (1-\lambda)Y^2$ is feasible for the minimization problem (iii′).

Part 4. We show that $C(u;\mathbf{p},\mathbf{y})$ convex in $u,\mathbf{y}\geq \mathbf{0}_N$ for every $\mathbf{p}\gg \mathbf{0}_N$ implies that the direct function $f(\mathbf{x},\mathbf{y}) \equiv \max_u\{u: C(u;\mathbf{p},\mathbf{y})\leq \mathbf{p}^T\mathbf{x}$ for every $\mathbf{p}\gg \mathbf{0}_M\}$ is concave in $\mathbf{x}\geq \mathbf{0}_M$, $\mathbf{y}\geq \mathbf{0}_N$. Let $\mathbf{x}^1\geq \mathbf{0}_M$, $\mathbf{x}^2\geq \mathbf{0}_M$, $\mathbf{y}^1\geq \mathbf{0}_N$, $\mathbf{y}^2\geq \mathbf{0}_N$, $0\leq\lambda\leq 1$, and define u^i, $i=1,2$, by

(iv) $f(\mathbf{x}^i,\mathbf{y}^i) \equiv \max_u\{u: C(u;\mathbf{p},\mathbf{y}^i)\leq \mathbf{p}^T\mathbf{x}^i \text{ for every } \mathbf{p}\gg \mathbf{0}_M\} \equiv u^i.$

Then

(iv′)
$$\begin{aligned} &f(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2,\ \lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2)\\ &\equiv \max_u\{u: C(u;\mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2)\leq \mathbf{p}^T(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2)\\ &\quad \text{for every } \mathbf{p}\gg \mathbf{0}_M\}\\ &\geq \lambda u^1 + (1-\lambda)u^2\\ &= \lambda f(\mathbf{x}^1,\mathbf{y}^1) + (1-\lambda)f(\mathbf{x}^2,\mathbf{y}^2), \end{aligned}$$

since by the convexity property of C, for every $\mathbf{p} \gg \mathbf{0}_M$, $C(\lambda u^1 + (1-\lambda)u^2;\ \mathbf{p},\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2) \le \lambda C(u^1;\mathbf{p},\mathbf{y}^1) + (1-\lambda)C(u^2;\mathbf{p},\mathbf{y}^2) \le \lambda\mathbf{p}^T\mathbf{x}^1 + (1-\lambda)\mathbf{p}^T\mathbf{x}^2$ [using (iv)], which means $\lambda u^1 + (1-\lambda)u^2$ is feasible for the maximization problem (iv′), and the solution to the maximization problem is equal to or greater than any feasible solution.

Note: If $C(u;\mathbf{p},\mathbf{y})$ is monotonically non-decreasing and continuous from below in u, then the set $L_{\mathbf{p}}(\mathbf{x},\mathbf{y}) \equiv \{u: C(u;\mathbf{p},\mathbf{y}) \le \mathbf{p}^T\mathbf{x}\}$ will be closed and bounded from above and thus the intersection $L(\mathbf{x},\mathbf{y}) \equiv \bigcap_{\mathbf{p} \gg \mathbf{0}_M} \{L_{\mathbf{p}}(\mathbf{x},\mathbf{y})\}$ will also be closed and bounded from above. Thus $f(\mathbf{x},\mathbf{y}) \equiv \max_u\{u: u \in L(\mathbf{x},\mathbf{y})\}$ is well-defined as a maximum.

Proof of Theorem 3.20

Part 1. Let $g(\mathbf{v},\mathbf{y})$ be convex in $\mathbf{v} \gg \mathbf{0}_M$ for $\mathbf{y} \ge \mathbf{0}_N$. Using the Fenchel closure operation [see Rockafellar (1970)], the extension of $g(\mathbf{v},\mathbf{y})$ to $\mathbf{v} \ge \mathbf{0}_M$ will also be convex. Let $\mathbf{x} \gg \mathbf{0}_M$, $Z^i > 0$, $0 \le \lambda \le 1$, and define $\mathbf{v}^i$, $i = 1,2$, by

(i) $f(\mathbf{x}/Z^i,\mathbf{y}) \equiv \min_{\mathbf{v}}\{g(\mathbf{v},\mathbf{y}): \mathbf{v}^T\mathbf{x}/Z^i \le 1, \mathbf{v} \ge \mathbf{0}_M\} \equiv g(\mathbf{v}^i,\mathbf{y})$.

Then

$$\begin{aligned} & f(\mathbf{x}/(\lambda Z^1 + (1-\lambda)Z^2,\mathbf{y}) \\ & \quad \equiv \min_{\mathbf{v}}\{g(\mathbf{v},\mathbf{y}): \mathbf{v}^T\mathbf{x}/(\lambda Z^1 + (1-\lambda)Z^2) \le 1, \mathbf{v} \ge \mathbf{0}_M\} \\ & \quad = \min_{\mathbf{v}}\{g(\mathbf{v},\mathbf{y}): \mathbf{v}^T\mathbf{x} \le \lambda Z^1 + (1-\lambda)Z^2, \mathbf{v} \ge \mathbf{0}_M\} \\ & \quad \le g(\lambda\mathbf{v}^1 + (1-\lambda)\mathbf{v}^2,\mathbf{y}) \end{aligned}$$

[since by (i), $(\lambda\mathbf{v}^1 + (1-\lambda)\mathbf{v}^2)^T\mathbf{x} \le \lambda Z^1 + (1-\lambda)Z^2$, and hence is feasible for the minimization problem],

$$\le \lambda g(\mathbf{v}^1,\mathbf{y}) + (1-\lambda)g(\mathbf{v}^2,\mathbf{y})$$

[using the convexity of g in v],

$$= \lambda f(\mathbf{x}/Z^1,\mathbf{y}) + (1-\lambda)f(\mathbf{x}/Z^2,\mathbf{y})$$

[using (i)].

Part 2. Let $f(\mathbf{x}/Z,\mathbf{y})$ be a convex function of the scalar variable $Z > 0$ for every $\mathbf{x} \gg \mathbf{0}_M$ for some $\mathbf{y} \ge \mathbf{0}_N$. Using the Fenchel closure operation, f

will have the same property for every $\mathbf{x} \geq \mathbf{0}_M$. Let $\mathbf{v} \gg \mathbf{0}_M$. Then

$$g(\mathbf{v},\mathbf{y}) \equiv \max_{\mathbf{z}}\{f(\mathbf{z},\mathbf{y}):\mathbf{v}^T\mathbf{z} \leq 1, \mathbf{z} \geq \mathbf{0}_M\}$$

$$= \max_{\mathbf{x}}\{f(\mathbf{x}/\mathbf{v}^T\mathbf{x},\mathbf{y}):\mathbf{x} \geq \mathbf{0}_M, \mathbf{x} \neq \mathbf{0}_M\}$$

[since f is non-decreasing in its arguments, $\mathbf{x} = \mathbf{0}_M$ is not a maximizer],

$$= \max_{\mathbf{x}}\{h_{\mathbf{x},\mathbf{y}}(\mathbf{v}):\mathbf{x} \in \boldsymbol{S}\}$$

[where $h_{\mathbf{x},\mathbf{y}}(\mathbf{v}) \equiv f(\mathbf{x}/\mathbf{v}^T\mathbf{x},\mathbf{y})$ is a convex function in $\mathbf{v} \gg \mathbf{0}_M$ for $\mathbf{x},\mathbf{y}$ fixed, and $\boldsymbol{S} \equiv \{\mathbf{x}:\mathbf{x} \geq \mathbf{0}_M, \mathbf{x} \neq \mathbf{0}_M\}$],

$$= \text{a convex function of } \mathbf{v}$$

[since the pointwise maximum over a set of convex functions is also a convex function – see Rockafellar (1970)].

Proof of Theorem 5.5

(i) In view of Lemmas 5.2 and 5.4, it remains to show that $c(\mathbf{p};\mathbf{w}) > 0$ if $\mathbf{p} \geq \mathbf{0}_I$ and $\mathbf{w} \gg \mathbf{0}_J$. If $\mathbf{p} \in \boldsymbol{P}$, then $c(\mathbf{p};\mathbf{w}) \equiv +\infty > 0$. If $\mathbf{p} \notin \boldsymbol{P}$, then $c(\mathbf{p};\mathbf{w}) \equiv \min_{\mathbf{x}}\{\mathbf{w}^T\mathbf{x}: V(\mathbf{p};\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} = \mathbf{w}^T\mathbf{x}^*$, say. Using the linear homogeneity (and continuity) of V in $\mathbf{x}$, we see that $\mathbf{x}^* > \mathbf{0}_J$ since $V(\mathbf{p};\mathbf{0}_J) = 0$. Since $\mathbf{w} \gg \mathbf{0}_J$, $\mathbf{w}^T\mathbf{x}^* > 0$.

(ii) We wish to show that for every scalar k, the set $\{(\mathbf{p};\mathbf{w}):c(\mathbf{p};\mathbf{w}) \geq k, \mathbf{p} \geq \mathbf{0}_I, \mathbf{w} \gg \mathbf{0}_J\}$ is closed in the set $\{(\mathbf{p};\mathbf{w}):\mathbf{p} \geq \mathbf{0}_I, \mathbf{w} \gg \mathbf{0}_J\}$. In view of the positivity of $c(\mathbf{p};\mathbf{w})$, we can restrict k to being non-negative. Let $k \geq 0$, $\mathbf{p}^n \geq \mathbf{0}_I$, $\mathbf{w}^n \gg \mathbf{0}_J$ for $n = 1,2,\ldots$, $\lim \mathbf{p}^n = \mathbf{p}^0 \geq \mathbf{0}_I$, $\lim \mathbf{w}^n = \mathbf{w}^0 \gg \mathbf{0}_J$ and $c(\mathbf{p}^n,\mathbf{w}^n) \geq k$ for each n. *Suppose* that $c(\mathbf{p}^0,\mathbf{w}^0) = k - \epsilon$ for some $\epsilon > 0$. Since $c(\mathbf{p}^0;\mathbf{w}^0)$ is finite, $\mathbf{p}^0 \notin \boldsymbol{P}$. Thus there exists $\mathbf{x}^0 \geq \mathbf{0}_J$ such that $c(\mathbf{p}^0;\mathbf{w}^0) = \mathbf{w}^{0T}\mathbf{x}^0 = k - \epsilon$, where $V(\mathbf{p}^0,\mathbf{x}^0) \geq 1$. In view of the continuity of V in p, we have for every $\delta > 0$, $V(\mathbf{p}^n;\mathbf{x}^0) \geq 1 - \delta$, for n large enough. Choose $\delta > 0$ to be such that $\delta < \min\{1,\epsilon/2k\}$. Therefore, for n large enough, $1 \leq V(\mathbf{p}^n;\mathbf{x}^0)/(1-\delta) = V(\mathbf{p}^n;\mathbf{x}^0/(1-\delta))$, using the linear homogeneity of V in $\mathbf{x}$. Since $\mathbf{w}^{nT}\mathbf{x}^0$ tends to $\mathbf{w}^{0T}\mathbf{x}^0$, for large enough n, we have $\mathbf{w}^{nT}\mathbf{x}^0 \leq k - \epsilon/2$. Thus for n large, $c(\mathbf{p}^n;\mathbf{w}^n) \equiv \min_{\mathbf{x}}\{\mathbf{w}^{nT}\mathbf{x}: V(\mathbf{p}^n;\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} \leq \mathbf{w}^{nT}\mathbf{x}^0/(1-\delta) \leq (k - \epsilon/2)/(1-\delta) < k$. This contradicts the hypothesis $c(\mathbf{p}^n;\mathbf{w}^n) \geq k$, and thus our supposition must be false and therefore we must have $c(p^0;w^0) \geq k$.

(iii) Let $\mathbf{p}^1 \geq \mathbf{0}_I$, $\mathbf{p}^2 \geq \mathbf{0}_I$, $\mathbf{w}^1 \gg \mathbf{0}_J$, $\mathbf{w}^2 \gg \mathbf{0}_J$, $k \geq 0$, $0 \leq \lambda \leq 1$. Assume that $c(\mathbf{p}^1;\mathbf{w}^1) \geq k$ and $c(\mathbf{p}^2;\mathbf{w}^2) \geq k$. We wish to show that $c(\lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^2;$

$\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)\geq k$. There are three cases to consider. Case (1); $\mathbf{p}^1\in \boldsymbol{P}$, $\mathbf{p}^2\in \boldsymbol{P}$. Since $\boldsymbol{P}$ is a convex set, $\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2\in \boldsymbol{P}$, and thus $c(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)=+\infty\geq k$. Case (2): $\mathbf{p}^1\notin \boldsymbol{P}$, $\mathbf{p}^2\notin \boldsymbol{P}$. If $\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2\in \boldsymbol{P}$, then $c(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)=+\infty\geq k$. Assume $\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2\notin \boldsymbol{P}$. If $k=0$, then the desired result follows using the positivity of c. Thus we assume $k>0$. We have $c(\mathbf{p}^1;\mathbf{w}^1)\equiv \min_{\mathbf{x}}\{\mathbf{w}^{1T}\mathbf{x}: V(\mathbf{p}^1;\mathbf{x})\geq 1,\quad \mathbf{x}\geq \mathbf{0}_J\}=\mathbf{w}^{1T}\mathbf{x}^1\geq k>0$ and $c(\mathbf{p}^2;\mathbf{w}^2)\equiv \min_{\mathbf{x}}\{\mathbf{w}^{2T}\mathbf{x}: V(\mathbf{p}^2;\mathbf{x})\geq 1,\ \mathbf{x}\geq \mathbf{0}_J\}=\mathbf{w}^{2T}\mathbf{x}^2\geq k>0$. Now if $\mathbf{x}\gg \mathbf{0}_J$, then there exists $\lambda^*>0$ such that $\mathbf{0}_J\leq \lambda^*\mathbf{1}\leq \mathbf{x}$. Therefore, $V(\mathbf{p}^1;\mathbf{x})\geq V(\mathbf{p}^1;\lambda^*\mathbf{1})$ [using 4.4(v)] $=\lambda^* V(\mathbf{p}^1;\mathbf{1})$ [using 4.4(iv)] >0 since $\mathbf{p}^1\notin \boldsymbol{P}$. Similarly, if $\mathbf{x}\gg \mathbf{0}_J$, then $V(\mathbf{p}^2;\mathbf{x})>0$. Thus if $\mathbf{x}\gg \mathbf{0}_J$, then $\mathbf{x}/V(\mathbf{p}^1;\mathbf{x})\geq \mathbf{0}_J$ and $V(\mathbf{p}^1;\mathbf{x}/V(\mathbf{p}^1;\mathbf{x}))=1$. Using the minimum nature of $\mathbf{w}^{1T}\mathbf{x}^1$, we have for every $\mathbf{x}\gg \mathbf{0}_J$, $\mathbf{w}^{1T}\mathbf{x}^1\leq \mathbf{w}^{1T}\mathbf{x}/V(\mathbf{p}^1;\mathbf{x})$ or $V(\mathbf{p}^1;\mathbf{x})\mathbf{w}^{1T}\mathbf{x}^1\leq \mathbf{w}^{1T}\mathbf{x}$ and using the continuity of V in $\mathbf{x}$ and the assumption that $\mathbf{w}^{1T}\mathbf{x}^1>0$ we find that $V(\mathbf{p}^1;\mathbf{x})\leq \mathbf{w}^{1T}\mathbf{x}/\mathbf{w}^{1T}\mathbf{x}^1$ for every $\mathbf{x}\geq \mathbf{0}_J$. Similarly, $V(\mathbf{p}^2;\mathbf{x})\leq \mathbf{w}^{2T}\mathbf{x}/\mathbf{w}^{2T}\mathbf{x}^2$ for every $\mathbf{x}\geq \mathbf{0}_J$. Let $\mathbf{x}\geq \mathbf{0}_J$. Then $V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{x})\leq \lambda V(\mathbf{p}^1;\mathbf{x})+(1-\lambda)V(\mathbf{p}^2;\mathbf{x})$ [using 4.4(iii)] $\leq \lambda\mathbf{w}^{1T}\mathbf{x}/\mathbf{w}^{1T}\mathbf{x}^1+(1-\lambda)\mathbf{w}^{2T}\mathbf{x}/\mathbf{w}^{2T}\mathbf{x}^2\leq \lambda\mathbf{w}^{1T}\mathbf{x}/k+(1-\lambda)\mathbf{w}^{2T}\mathbf{x}/k$, since $\mathbf{w}^{1T}\mathbf{x}^1\geq k$ and $\mathbf{w}^{2T}\mathbf{x}^2\geq k$. Thus if $\mathbf{x}\geq \mathbf{0}_J$ and $V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{x})\geq 1$, then $(\lambda\mathbf{w}^{1T}\mathbf{x}+(1-\lambda)\mathbf{w}^{2T}\mathbf{x})/k>1$. Now $c(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)\equiv \min_{\mathbf{x}}\{(\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)^T\mathbf{x}: V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{x})\geq 1,\quad \mathbf{x}\geq \mathbf{0}_J\}\geq \min_{\mathbf{x}}\{(\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)^T\mathbf{x}:(\lambda\mathbf{w}^{1T}\mathbf{x}+(1-\lambda)\mathbf{w}^{2T}\mathbf{x})/k\geq 1,\ \mathbf{x}\geq \mathbf{0}_J\}$ [since in general the set of feasible x's in the second minimization problem is larger] $=\min_{\mathbf{x}}\{(\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)^T\mathbf{x}:(\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)^T\mathbf{x}\geq k;\ \mathbf{x}\geq \mathbf{0}_J\}$ [since $k>0$] $=k$. Case (3): $\mathbf{p}^1\notin \boldsymbol{P}$ and $\mathbf{p}^2\in \boldsymbol{P}$. Without loss of generality, we can take $\mathbf{p}^2$ belonging to the boundary of $\boldsymbol{P}$. Using the results of case (2) above, we need only show that as λ tends to zero, $c(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)$ tends to plus infinity. Let $\mathbf{p}^1\geq \mathbf{0}_I$, $\mathbf{p}^2\geq \mathbf{0}_I$, $\mathbf{w}^1\gg \mathbf{0}_J$, $\mathbf{w}^2\gg \mathbf{0}_J$, and let $\beta>0$ be the minimum component of the vectors $\mathbf{w}^1$ and $\mathbf{w}^2$. For $0\leq\lambda\leq 1$, define $f(\lambda)\equiv V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{1})$. By hypothesis, $f(0)=0$, but $f(\lambda)>0$ for $0<\lambda\leq 1$. Since by (i) and (iii) of 4.4, $V(\mathbf{p};\mathbf{1})$ is a non-negative, convex function of $\mathbf{p}$, we see that $f(\lambda)$ is a non-decreasing, convex function of λ for $0\leq\lambda\leq 1$, such that $\lim_{\lambda\to 0}\{f(\lambda):\lambda>0\}=0$. Now if $0<\lambda\leq 1$, we have $V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \mathbf{1}/f(\lambda))=1$. Thus if $\mathbf{0}_J\leq \mathbf{x}\ll \mathbf{1}/f(\lambda)$, then using (iv) and (v) of 4.4, we find $V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{x})<1$. Therefore, for $0<\lambda\leq 1$, $c(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\ \lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)\equiv \min_{\mathbf{x}}\{(\lambda\mathbf{w}^1+(1-\lambda)\mathbf{w}^2)^T\mathbf{x}: V(\lambda\mathbf{p}^1+(1-\lambda)\mathbf{p}^2;\mathbf{x})\geq 1;\ \ \mathbf{x}\geq \mathbf{0}_J\}\geq \beta/f(\lambda)$, and the last term tends monotonically to plus infinity as λ tends to zero.

(iv) Let $\mathbf{p}\geq \mathbf{0}_I$, $\mathbf{w}\gg \mathbf{0}_J$ and $\lambda>0$. Then $c(\lambda\mathbf{p};\mathbf{w})\equiv \min_{\mathbf{x}}\{\mathbf{w}^T\mathbf{x}: V(\lambda\mathbf{p};\mathbf{x})\geq 1,\ \ \mathbf{x}\geq \mathbf{0}_J\}=\min_{\mathbf{x}}\{\mathbf{w}^T\mathbf{x}: V(\mathbf{p};\lambda\mathbf{x})\geq 1,\ \ \mathbf{x}\geq \mathbf{0}_J\}$ [using (ii) and 4.4(iv)] $=\lambda^{-1}\min_{\mathbf{x}}\{\mathbf{w}^T\lambda\mathbf{x}: V(\mathbf{p};\lambda\mathbf{x})\geq 1,\ \lambda\mathbf{x}\geq \mathbf{0}_J\}=\lambda^{-1}c(\mathbf{p};\mathbf{w})$.

(v) Let $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \gg \mathbf{0}_J$, $\lambda > 0$. Then $c(\mathbf{p};\lambda\mathbf{w}) \equiv \min_{\mathbf{x}}\{\lambda\mathbf{w}^T\mathbf{x}: V(\mathbf{p};\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} = \lambda c(\mathbf{p};\mathbf{w})$.

(vi) Let $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w}^2 \geq \mathbf{w}^1 \gg \mathbf{0}_J$. Then $c(\mathbf{p};\mathbf{w}^2) \equiv \min_{\mathbf{x}}\{\mathbf{w}^{2T}\mathbf{x}: V(\mathbf{p};\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} = \mathbf{w}^{2T}\mathbf{x}^2 = \mathbf{w}^{1T}\mathbf{x}^2 + (\mathbf{w}^2 - \mathbf{w}^1)^T\mathbf{x}^2 \geq \mathbf{w}^{1T}\mathbf{x}^2$ [since $\mathbf{w}^2 - \mathbf{w}^1 \geq \mathbf{0}_J$ and $\mathbf{x}^2 \geq \mathbf{0}_J$] $\geq \min_{\mathbf{x}}\{\mathbf{w}^{1T}\mathbf{x}: V(\mathbf{p};\mathbf{x}) \geq 1, \mathbf{x} \geq \mathbf{0}_J\} = c(\mathbf{p};\mathbf{w}^1)$.

Proof of Theorem 5.8

(i) $V^*(\mathbf{p};\mathbf{x}) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x} \leq 1\}$ for $\mathbf{p} \geq \mathbf{0}_I$ and $\mathbf{x} \gg \mathbf{0}_J$ is well-defined as a minimum, since $1/c(\mathbf{p};\mathbf{w})$ is continuous from below in $\mathbf{w}$, non-negative and finite for at least one value of $\mathbf{w}$ and the minimum of such a function over the closed, bounded set $\{\mathbf{w}:\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x} \leq 1\}$ exists. (Recall Theorem 1.5.) We note that since $c(\mathbf{p};\mathbf{w})$ is non-negative, continuous from above, and quasi-concave for $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{w} \geq \mathbf{0}_J$, $1/c(\mathbf{p};\mathbf{w})$ will be non-negative, continuous from below, and quasi-convex over the same domain.

(ii) Let $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{x} \gg \mathbf{0}_J$ and $\lambda > 0$. Then $V^*(\lambda\mathbf{p};\mathbf{x}) \equiv \min_{\mathbf{w}}\{1/c(\lambda\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x} \leq 1 = \min_{\mathbf{w}}\{1/\lambda^{-1}c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x} \leq 1\}$ [using 5.6(iv)] $= \lambda V^*(\mathbf{p};\mathbf{x})$.

(iii) We first show $V^*(\mathbf{p};\mathbf{x})$ is quasi-convex in $\mathbf{p}$ for fixed $\mathbf{x}$. Let $\mathbf{x} \gg \mathbf{0}_J$, $\mathbf{p}^1 \geq \mathbf{0}_I$, $\mathbf{p}^2 \geq \mathbf{0}_I$, $0 \leq \lambda \leq 1$ and $k \geq 0$. Let $k \geq V^*(\mathbf{p}^i;\mathbf{x}) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p}^i;\mathbf{w}):\mathbf{w} \geq \mathbf{0}, \mathbf{w}^T\mathbf{x} \leq 1\} = 1/c(\mathbf{p}^i;\mathbf{w}^i)$, say. Therefore $c(\mathbf{p}^1;\mathbf{w}^1) \geq 1/k$, $c(\mathbf{p}^2;\mathbf{w}^2) \geq 1/k$, and by 5.6(iii), $c(\lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^2; \lambda\mathbf{w}^1 + (1-\lambda)\mathbf{w}^2) \geq 1/k$ also. Thus $V^*(\lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^2;\mathbf{x}) \equiv \min_{\mathbf{w}}\{1/c(\lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^2;\mathbf{w}):\mathbf{w} \geq \mathbf{0}, \mathbf{w}^T\mathbf{x} \leq 1\} \leq 1/c(\lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^2; \lambda\mathbf{w}^1 + (1-\lambda)\mathbf{w}^2)$ [since $\lambda\mathbf{w}^1 + (1-\lambda)\mathbf{w}^2 \geq 0$ and $(\lambda\mathbf{w}^1 + (1-\lambda)\mathbf{w}^2)^T\mathbf{x} \leq 1] \leq k$. Finally since $V^*(\mathbf{p};\mathbf{x})$ is non-negative, quasi-convex and (positively) homogeneous of degree one over the set $\mathbf{p} \geq \mathbf{0}$ for a fixed $\mathbf{x}$, it can be seen [see Berge (1963, p. 208)] that $V^*(\mathbf{p};\mathbf{x})$ is also a convex function in $\mathbf{p}$ for fixed $\mathbf{x}$. With respect to the continuity of $V^*(\mathbf{p};\mathbf{x})$ in $\mathbf{p}$, we note that the convexity of $V^*(\mathbf{p};\mathbf{x})$ in $\mathbf{p}$ will imply that $V^*(\mathbf{p};\mathbf{x})$ is continuous from above in $\mathbf{p} \geq \mathbf{0}_I$ for fixed $\mathbf{x}$ [see Rockafellar (1970, p. 84)]. Now $-V^*(\mathbf{p};\mathbf{x}) = -\min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\mathbf{x} \leq 1\} = \max_{\mathbf{w}}\{-1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}, \mathbf{w}^T\mathbf{x} \leq 1\}$ and $-V^*(\mathbf{p};\mathbf{x})$ will be continuous from above by the Upper Semicontinuity Maximum Theorem 1.8 and thus $V^*(\mathbf{p};\mathbf{x})$ will be continuous from below in $\mathbf{p}$ for fixed $\mathbf{x} \gg \mathbf{0}_J$. Thus $V^*(\mathbf{p};\mathbf{x})$ will be continuous in $\mathbf{p}$.

(iv) Let $\mathbf{p} \geq \mathbf{0}_I$, $\mathbf{x} \gg \mathbf{0}_J$ and $\lambda > 0$. Then $V^*(\mathbf{p};\lambda\mathbf{x}) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J, \mathbf{w}^T\lambda\mathbf{x} \geq 1\} = \min_{\mathbf{w}}\{\lambda/c(\mathbf{p};\lambda\mathbf{w}):\lambda\mathbf{w} \geq \mathbf{0}, (\lambda\mathbf{w})^T\mathbf{x} \leq 1\}$ [using 5.6(v)] $= \min_{\mathbf{w}^*}\{1/c(\mathbf{p};\mathbf{w}^*):\mathbf{w}^* \geq \mathbf{0}, \mathbf{w}^{*T}\mathbf{x} \leq 1\} = \lambda V^*(\mathbf{p};\mathbf{x})$.

(v) Let $\mathbf{x}^2 \geq \mathbf{x}^1 \gg \mathbf{0}_J$, $\mathbf{p} \geq \mathbf{0}_I$. Then $V^*(\mathbf{p};\mathbf{x}^2) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \geq \mathbf{0}_J$,

$\mathbf{w}^T\mathbf{x}^2 \le 1\} \ge \min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^1 \le 1\}$ [since the set of feasible w's has not decreased, the minimum cannot increase] $= V^*(\mathbf{p};\mathbf{x}^1)$.

(vi) If $\mathbf{p} \in \boldsymbol{P}$, then $c(\mathbf{p};\mathbf{w}) = +\infty$ for all $\mathbf{w} \ge \mathbf{0}_J$, and thus $V^*(\mathbf{p};\mathbf{x}) = 0$ for all $\mathbf{x} \gg \mathbf{0}_J$, and $V^*(\mathbf{p};\mathbf{x})$ is a concave function in $\mathbf{x}$. Let $\mathbf{p}^* \ge \mathbf{0}_I$, $\mathbf{p}^* \notin \boldsymbol{P}$. Then $V^*(\mathbf{p}^*;\mathbf{x})$ will be a concave function in $\mathbf{x} \gg \mathbf{0}_J$ if we can show that $V^*(\mathbf{p}^*;\mathbf{x})$ is a positive, linearly homogeneous, quasi-concave function in $\mathbf{x} \gg \mathbf{0}_J$. [See Berge (1963, p. 208) or Newman (1969, p. 300).] Since $\mathbf{p}^* \notin \boldsymbol{P}$, we have $c(\mathbf{p}^*;\mathbf{w}) > 0$ and finite for $\mathbf{w} \gg \mathbf{0}_J$. Thus if $\mathbf{x} \gg \mathbf{0}_J$, $V^*(\mathbf{p}^*;\mathbf{x}) > 0$. Linear homogeneity of $V^*(\mathbf{p}^*;\mathbf{x})$ in $\mathbf{x}$ follows from (iv) above, and thus it remains to show that $V^*(\mathbf{p}^*;\mathbf{x})$ is quasi-concave in $\mathbf{x}$. Let $\mathbf{x}^1 \gg \mathbf{0}_J$, $\mathbf{x}^2 \gg \mathbf{0}_J$, $0 \le \lambda \le 1$, $k > 0$, and suppose that $V^*(\mathbf{p}^*;\mathbf{x}^1) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p}^*;\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^1 \le 1\} = 1/c(\mathbf{p}^*;\mathbf{w}^1) \ge k$, and $V^*(\mathbf{p}^*;\mathbf{x}^2) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p}^*;\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^2 \le 1\} = 1/c(\mathbf{p}^*;\mathbf{w}^2) \ge k$. $V^*(\mathbf{p}^*;\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p}^*;\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2) \le 1\} = 1/c(\mathbf{p}^*;\mathbf{w}^*)$, say. Suppose $1/c(\mathbf{p}^*;\mathbf{w}^*) < k$. Then since $\mathbf{w}^{*T}(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2) \le 1$, we must have either $\mathbf{w}^{*T}\mathbf{x}^1 \le 1$ or $\mathbf{w}^{*T}\mathbf{x}^2 \le 1$ (or both). If $\mathbf{w}^{*T}\mathbf{x}^1 \le 1$, then $V^*(\mathbf{p}^*;\mathbf{x}^1) \le 1/c(\mathbf{p}^*;\mathbf{w}^*) < k$, which contradicts $V^*(\mathbf{p}^*;\mathbf{x}^1) \ge k$. Similarly, if $\mathbf{w}^{*T}\mathbf{x}^2 \le 1$, then we get a contradiction to $V^*(\mathbf{p}^*;\mathbf{x}^2) \ge k$. Thus our supposition $1/c(\mathbf{p}^*;\mathbf{w}^*) < k$ must be false, and $V^*(\mathbf{p}^*;\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2) \ge k$. We have established that for a fixed $\mathbf{p} \ge \mathbf{0}_I$, $V^*(\mathbf{p};\mathbf{x})$ is a concave function in $\mathbf{x}$ for $\mathbf{x} \gg \mathbf{0}_J$, and thus $V^*(\mathbf{p};\mathbf{x})$ is also continuous in $\mathbf{x} \gg \mathbf{0}_J$ for fixed $\mathbf{p}$.

To complete the proof of Theorem 5.8, assume that a value-added function V satisfied Conditions II given by 4.4, that c was defined by equation 5.3 and that V^* was defined by equation 5.7. We wish to show that for every $\mathbf{p} \ge \mathbf{0}_I$, $\mathbf{x}^* \gg \mathbf{0}_J$, $V^*(\mathbf{p};\mathbf{x}^*) = V(\mathbf{p};\mathbf{x}^*)$. If $\mathbf{p} \in \boldsymbol{P}$ defined by equation 5.1, it is easy to see that $V^*(\mathbf{p};\mathbf{x}^*) = V(\mathbf{p};\mathbf{x}^*) = 0$. Assume $\mathbf{p} \notin \boldsymbol{P}$. Since $\mathbf{x}^* \gg \mathbf{0}_J$, $V(\mathbf{p};\mathbf{x}^*) = v > 0$. Define the value-added production possibilities set $\boldsymbol{L}(v;\mathbf{p}) \equiv \{\mathbf{x}: V(\mathbf{p};\mathbf{x}) \ge v;\ \mathbf{x} \ge \mathbf{0}_J\}$. By properties (iv) and (vi) of 4.4, we see that $\mathbf{x}^*$ is a boundary point of the closed convex set $\boldsymbol{L}(v;\mathbf{p})$ and thus $\boldsymbol{L}(v;\mathbf{p})$ will have at least one supporting hyperplane $\mathbf{w}^* \ne \mathbf{0}$ at the point $\mathbf{x}^*$, i.e., $\mathbf{w}^{*T}\mathbf{x}^* \le \mathbf{w}^{*T}\mathbf{x}$ for every $\mathbf{x} \in \boldsymbol{L}(v;\mathbf{p})$. Property 4.4(v) will imply that $\mathbf{w}^* > \mathbf{0}_J$. Thus $\mathbf{w}^{*T}(\mathbf{x}^*/v) = c(\mathbf{p};\mathbf{w}^*) \equiv \min_{\mathbf{x}}\{\mathbf{w}^{*T}\mathbf{x}: V(\mathbf{p};\mathbf{x}) \ge 1,\ \mathbf{x} \ge \mathbf{0}_J\}$ and $c(\mathbf{p};\mathbf{w}) \le \mathbf{w}^T(\mathbf{x}^*/v)$ for every $\mathbf{w} \ge \mathbf{0}$ since $V(\mathbf{p};\mathbf{x}^*/v) \ge 1$. Now $V^*(\mathbf{p};\mathbf{x}^*) \equiv \min_{\mathbf{w}}\{1/c(\mathbf{p};\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^* \le 1\} = 1/\max_{\mathbf{w}}\{c(\mathbf{p};\mathbf{w}):\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^* \le 1\} = v/\max_{\mathbf{w}}\{c(\mathbf{p};\mathbf{w})v:\mathbf{w} \ge \mathbf{0}_J,\ \mathbf{w}^T\mathbf{x}^* \le 1$ [since $v > 0$] $= v$ [since $c(\mathbf{p};\mathbf{w})v \le \mathbf{w}^T\mathbf{x}^*$ for every $\mathbf{w} \ge \mathbf{0}_J$ with equality for $\mathbf{w} = \mathbf{w}^*$] $= V(\mathbf{p};\mathbf{x}^*)$ by the definition of v.

Q.E.D.