Chapter II.2

THE GENERAL LINEAR PROFIT FUNCTION

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1. Introduction

The classical competitive firm is assumed to face exogenously determined technological possibilities and choose variable inputs and outputs to maximize profits at exogenous competitive market prices. This behavior can be summarized in a *restricted profit function* specifying maximum profit as a function of the exogenous variables, market prices and parameters specifying technological possibilities. By varying the interpretation of commodities and parameters, one can formulate as special cases of this general model the problems of cost minimization, revenue maximization, intertemporal operation of the firm and operation of the firm under uncertainty. In Chapter I.1, the author has given a detailed discussion of properties and possible applications of the restricted profit function.

The practical advantage of formulating a model of the competitive firm in terms of a restricted profit function lies in the computationally simple relationship between this function and the derived demand and supply functions which form the basis for comparative analysis or econometric estimation; namely, the net supply functions can be computed as partial derivatives of the restricted profit function with respect to market prices. Judicious choice of a functional form for the

*The concepts of a linear-in-parameters cost function and the second-order approximation property are due to E. Diewert, and I am indebted to him and to M. Fuss for many useful discussions. This research was supported by NSF Grant GS-2345.

restricted profit function can yield net supply systems which embody economic phenomena of interest and which are convenient for statistical analysis. This chapter introduces a class of *general linear profit functions* which should provide useful functional forms from the standpoint of both these criteria. These functional forms have the properties:

(a) They are linear in the underlying parameters of the production process, making it possible to estimate the net supply system by multivariate linear regression techniques and formulate economic hypotheses as linear restrictions on this system.
(b) They satisfy globally (i.e., for all positive market prices) the criteria for a function to be the restricted profit function associated with some technology.
(c) They can approximate a large class of restricted profit functions (e.g., those satisfying a gross substitutes property[1]) up to the second order at any specified argument, thus agreeing on net supply quantities and price elasticities at this argument.

An additional advantage of these functional forms is that aggregation over firms with common technologies "carries past" the unknown parameters, permitting a simple theory of aggregation and estimation from aggregate data. The general linear profit function is an extension of the generalized Leontief cost function introduced by Diewert (1971), and can reduce to his cost function in the case of cost minimization for fixed output.

2. The Basic Model

Consider a firm facing competitive markets in N commodities, indexed $n = 1,\ldots,N$, with a commodity price vector $\mathbf{p} = (p_1,\ldots,p_N)$. A production plan for the firm is a vector $\mathbf{x} = (x_1,\ldots,x_N)$ with x_n interpreted as the net supply (or, for compactness, netput) of commodity n, negative if the commodity is an input and positive if it is an output. The profit associated with a production plan $\mathbf{x}$ is $\pi = \mathbf{p}\cdot\mathbf{x} = p_1x_1 + \cdots + p_nx_n$. The technological possibilities of the firm can be described by a set $\mathbf{T}$ of possible production plans. This set will in general depend on variables

[1]A technology has the gross substitutes property if the optimal net supply of each commodity is non-increasing in the price of every other commodity.

exogenous to the firm, as for example the state of technical progress, fixed outputs in the case of cost minimization, and fixed capital inputs in the case of short-run profit maximization. To simplify notation, we leave to the reader the task of introducing this dependence explicitly in the formulae below.

Define the production possibility set $\mathbf{T}$ to be *regular* if it is non-empty and closed and satisfies the free disposal property that $\mathbf{x} \in \mathbf{T}$ and $\mathbf{x}' \leqq \mathbf{x}$ implies $\mathbf{x}' \in \mathbf{T}$. Define $\mathbf{T}$ to be *asymptotically irreversible* (or semi-bounded) if there is a bound on the vectors of production plans $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^n \in \mathbf{T}$ satisfying $\sum_{i=0}^{N} \mathbf{x}^i = 0$. This condition excludes the possibility of a "perpetual motion" production process of unbounded "amplitude", and will hold if there are some non-producible commodities which are essential inputs to production.

The *restricted profit function* of the firm with technology $\mathbf{T}$ is

$$\pi = \Pi(\mathbf{p}) = \sup_{\mathbf{x} \in \mathbf{T}} \mathbf{p} \cdot \mathbf{x}, \tag{1}$$

and gives the least upper bound (possibly $+\infty$) on the level of profits attainable at price vector $\mathbf{p}$. Let $(\mathbf{dom}\, \Pi)$ denote the set of price vectors for which $\Pi(\mathbf{p})$ is finite.

An extended real-valued function $Q: \mathbf{E}^n \to [-\infty, +\infty]$ is said to be of *type* **RP** if it satisfies

(1) the set $(\mathbf{dom}\, Q)$ on which Q is finite is a convex cone with a non-empty interior which is contained in the non-negative orthant of $\mathbf{E}^n$; and
(2) Q is a convex conical closed[2] function on $(\mathbf{dom}\, Q)$.

A basic duality between production possibility sets and restricted profit functions is established in the following theorem, proved in Chapter I.1, Lemmas 11, 23.

Theorem 1. If $\mathbf{T}$ is a regular asymptotically irreversible production possibility set, then the restricted profit function Π defined by (1) is of type **RP**. Alternatively, if Π is a function of type **RP**, then the set

$$\mathbf{T}^* = \{\mathbf{x} \in \mathbf{E}^N \mid \mathbf{p} \cdot \mathbf{x} \leqq \Pi(\mathbf{p}) \quad \text{for} \quad \mathbf{p} \in \mathbf{E}^N\} \tag{2}$$

is a regular asymptotically irreversible convex production possibility

[2] A function Q is *convex* if $Q(\mathbf{p})$, $Q(\mathbf{p}') < +\infty$, $0 < \theta < 1$ implies $Q(\theta \mathbf{p} + (1-\theta)\mathbf{p}') \leqq \theta Q(\mathbf{p}) + (1-\theta) Q(\mathbf{p}')$; *conical* if $Q(\lambda \mathbf{p}) = \lambda Q(\mathbf{p})$ for $\lambda > 0$; and *closed* if the set $(\mathbf{epi}\, Q) = \{(\mathbf{p}, q) \mid q \geqq Q(\mathbf{p})\}$ is closed.

set. In particular, if the function Π in (2) is the restricted profit function of a regular asymptotically irreversible production possibility set $\mathbf{T}$, then $\mathbf{T}^*$ is the closed convex hull of $\mathbf{T}$. The mappings (1) and (2) are mutually inverse between the family of regular asymptotically irreversible convex production possibility sets and the family of functions of type **RP**; e.g., applying the mapping (2) to a function Π of type **RP** and then applying the mapping (1) to the resulting set $\mathbf{T}^*$ returns the function Π.

A second basic property of the restricted profit function is the *derivative property*, proved in Chapter I.1, Lemmas 17–19.

Theorem 2. Consider a function Π of type **RP**. Π is differentiable at $\mathbf{p}'$ in the interior of $\mathbf{dom}\,\Pi$ if and only if in the technology $\mathbf{T}^*$ given by (2), there exists a unique vector $\mathbf{x}' \in \mathbf{T}^*$ at which $\mathbf{p}'\cdot\mathbf{x}$ is maximized on $\mathbf{T}^*$, in which case $\Pi_{\mathbf{p}}(\mathbf{p}') = \mathbf{x}'$.

In analyzing the general linear profit function below, we shall use on a function $Q:\mathbf{E}^n \to [-\infty,+\infty]$ of type **RP** the condition **C2** that Q be twice continuously differentiable with a Hessian of rank $n-1$ on the interior of ($\mathbf{dom}\,Q$), and the condition **FP** that ($\mathbf{dom}\,Q$) contain the positive orthant. Condition **C2** implies the dual technology of Q given by equation (2) is strictly convex (as viewed from the positive orthant of $\mathbf{E}^n$) with a "specific curvature" which is bounded positive. Condition **FP** implies that as the scale of production becomes large, the set of possible activities in the dual technology shrinks to the set of disposal activities, i.e., the asymptotic cone of the dual technology is the non-positive orthant. These duality implications are discussed in detail in Chapter I.1, Lemma 12, Theorem 26.

3. General Linear Profit Functions

A function $\Pi(\mathbf{p};\boldsymbol{\alpha})$ which is linear in a vector of underlying parameters $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ can be written in the form

$$\Pi(\mathbf{p};\boldsymbol{\alpha}) = \sum_{m=1}^{M} \alpha_m Q^m(\mathbf{p}). \tag{3}$$

where Q^m is a numerical function. Further, we can always standardize the parameter specification so that $\boldsymbol{\alpha}$ is restricted to be non-negative (i.e., we can first write each bivalent parameter as the difference of its positive and negative parts, and then re-define the Q function associated with each negative parameter to absorb its sign). This convention will be imposed hereafter in discussion of equation (3) unless explicitly assumed otherwise.

If the function $\Pi(\mathbf{p};\boldsymbol{\alpha})$ in equation (3) is of type **RP** for all non-negative $\boldsymbol{\alpha}$, then clearly each function Q^m is of type **RP** and $\bigcap_{m=1}^{M}(\mathbf{dom}\ Q^m)$ has a non-empty interior.[3] Conversely, it is an elementary property of convex functions that if each function Q^m is of type **RP** and if $\bigcap_{m=1}^{M}(\mathbf{dom}\ Q^m)$ has a non-empty interior, then Π given by equation (3) for any non-negative vector $\boldsymbol{\alpha}$ is of type **RP**.

A function Π in equation (3) which is of type **RP** for all non-negative $\boldsymbol{\alpha}$ will be termed a *general linear profit form*. This form can be specialized for econometric purposes by choosing specific numerical functions Q^m. To aid computation and interpretation it is convenient to take each function Q^m to depend on a small subset of the commodity prices. If each Q^m depends on a single price, then it is linear and the resulting linear profit function in equation (3) is dual to a pure fixed coefficients Leontief technology. The next case with the Q^m depending on pairs of commodity prices yields a variety of useful functional forms corresponding to a fairly broad class of technologies. Rewrite equation (3) by indexing over pairs of commodity prices as

$$\Pi(\mathbf{p};\boldsymbol{\alpha}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij} p_i Q^{ij}(p_j/p_i), \tag{4}$$

where the α_{ij} are non-negative parameters for $i \neq j$ with $\alpha_{ij} = \alpha_{ji}$, the Q^{ij} are closed convex functions of a positive real variable (implying $p_i Q^{ij}(p_j/p_i)$ is of type **RP** and satisfies condition **FP**), and the diagonal parameters α_{ii} are unrestricted in sign. One case of equation (4) is a version of Diewert's (1971) generalized Leontief form,

$$\Pi(\mathbf{p};\boldsymbol{\alpha}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij}(-(p_i p_j)^{1/2}); \tag{5}$$

others can be obtained by substituting for Q^{ij} in equation (4) some combination of the standard numerical forms for convex functions of one variable given in the first column of Table 1. When the functions Q^{ij} in (4) are differentiable, Theorem 2 implies the existence of an optimal

[3]Taking $\alpha_m > 0$, $\alpha_l = 0$ for $l \neq m$ implies $\Pi(\mathbf{p};\boldsymbol{\alpha}) = Q^m(\mathbf{p})$, so that Q^m is of type **RP**. Taking $\boldsymbol{\alpha}$ strictly positive implies $\mathbf{dom}\ \Pi(\cdot;\boldsymbol{\alpha}) = \bigcap_{m=1}^{M}(\mathbf{dom}\ Q^m)$.

TABLE 1

Convex functions of a positive real variable (θ and δ are numerical parameters).

	$Q^{ij}(r)$	Q^{ij}_r	$Q^{ij} - rQ^{ij}_r$	Q^{ij}_{rr}	Dual technology $\mathbf{T}^{ij}$ consisting of the points $\mathbf{x} \in \mathbf{E}^N$ with $x_k = 0$ for $k \neq i,j$ and x_i, x_j below, plus points obtained by free disposal
(1)	r^θ $\theta > 1$ or $\theta < 0$	$\theta r^{\theta-1}$	$(1-\theta)r^\theta$	$\theta(\theta-1)r^{\theta-2}$	$\left(\frac{x_i}{1-\theta}\right)^{1-\theta}\left(\frac{x_j}{\theta}\right)^\theta = 1$ $\frac{x_i}{1-\theta} \geq 0, \frac{x_j}{\theta} \geq 0$
(2)	$-r^\theta$ $0 < \theta < 1$	$-\theta r^{\theta-1}$	$-(1-\theta)r^\theta$	$-\theta(\theta-1)r^{\theta-2}$	$\left(\frac{-x_i}{1-\theta}\right)^{1-\theta}\left(\frac{-x_j}{\theta}\right)^\theta = 1$ $x_i, x_j \leq 0$
(3)	$[1-\delta+\delta r^{1+\sigma}]^{1/(1+\sigma)}$ $\sigma > 0$ $0 < \delta < 1$	$\delta r^\sigma (Q^{ij})^{-\sigma}$	$(1-\delta)(Q^{ij})^{-\sigma}$	$\delta(1-\delta)\sigma r^{\sigma-1}(Q^{ij})^{-2\sigma-1}$	$[(1-\delta)^{-1/\sigma}x_i^{1+1/\sigma} + \delta^{1/\sigma}x_j^{1+1/\sigma}] = 1$ $x_i, x_j \geq 0$

(4)	$-[1-\delta+\delta r^{1-\sigma}]^{1/(1-\sigma)}$ $\sigma>0,\ \sigma\neq 1$ $0<\delta<1$	$-\delta r^{-\sigma}(-Q^{ij})^{\sigma}$	$-(1-\delta)(-Q^{ij})^{\sigma}$	$-\delta(1-\delta)\sigma r^{-\sigma-1}(-Q^{ij})^{2\sigma-1}$	$[(1-\delta)^{1/\sigma}(-x_i)^{1-1/\sigma}+\delta^{1/\sigma}(-x_j)^{1-1/\sigma}]=1$ $x_i, x_j \leq 0$
(5)	$e^{\theta r}$ $\theta\neq 0$	$\theta e^{\theta r}$	$(1-r\theta)e^{\theta r}$	$\theta^2 e^{\theta r}$	$x_i=\left(\frac{x_j}{\theta}\right)\left[1-\log\left(\frac{x_j}{\theta}\right)\right]$ $\frac{x_j}{\theta}>0$
(6)	$-\log(\theta+r)$ $\theta\geq 0$	$-(\theta+r)^{-1}$	$-\log(\theta+r)+\frac{r}{\theta+r}$	$(\theta+r)^{-2}$	$x_i=\log(-x_j)+(1+\theta x_j)$ $-\theta^{-1}\leq x_j<0$
(7)	$-\frac{r}{\theta+r}$ $\theta>0$	$-\frac{\theta}{(\theta+r)^2}$	$-\frac{r^2}{(\theta+r)^2}$	$\frac{2\theta}{(\theta+r)^3}$	$x_j=\frac{x_i}{\theta}(-1+(-x_i)^{-1/2})^2$ $-1\leq x_i\leq 0$
(8)	$\left(r+\frac{\theta}{r}\right)$ $\theta>0$	$1-\frac{\theta}{r^2}$	$2\frac{\theta}{r}$	$\frac{\theta}{r^3}$	$x_j=1-\frac{x_i^2}{4\theta}$ $x_i\geq 0$

production plan $\hat{\mathbf{x}}(\mathbf{p})$ for positive p satisfying

$$\hat{x}_k(\mathbf{p}) = \frac{\partial \Pi}{\partial p_k} = \sum_{j=1}^{N} \alpha_{kj} \left[Q^{kj}\left(\frac{p_j}{p_k}\right) - \frac{p_j}{p_k} Q_r^{kj}\left(\frac{p_j}{p_k}\right) + Q_r^{jk}\left(\frac{p_k}{p_j}\right) \right], \tag{6}$$

and

$$\frac{\partial \hat{x}_k(\mathbf{p})}{\partial p_l} = \frac{\partial^2 \Pi}{\partial p_k \partial p_l} = -\alpha_{kl} \left[\frac{p_l}{p_k^2} Q_{rr}^{kl}\left(\frac{p_l}{p_k}\right) + \frac{p_k}{p_l^2} Q_{rr}^{lk}\left(\frac{p_k}{p_l}\right) \right], \tag{7}$$

for $k \neq l$, where Q_r^{ij} and Q_{rr}^{ij} denote the first and second derivatives, respectively, of the function Q^{ij}. Since the expressions in brackets in equations (6) and (7) are numerical functions, these formulae allow application of multivariate regression analysis to estimate the net supply system.

A technology is said to have the gross substitutes (**GS**) property if the optimal net supply of each commodity k is non-increasing in the price of every other commodity. This property corresponds to the "normal" case where all outputs are substitutes (e.g., the quantity of one falls when the price and quantity of a second rises), all inputs are non-regressive in the production of outputs (e.g., each input quantity rises when the price and quantity of an output rise), and all inputs are strong substitutes (e.g., an increase in the price of one input leads to substitution of a second input which is sufficient to offset the tendency of an input price increase to reduce output quantity, and thus input quantities[4]). When the restricted profit function Π of the technology has the differentiability property **C2**, the gross substitutes property can be defined as the condition $\partial^2\Pi/\partial p_k \partial p_l \leqq 0$ for $k \neq l$. A profit function of two prices must satisfy **GS**, and a sum of functions satisfying **GS** must again have this property. Thus, the linear profit form (4) has property **GS**; this is also clear from the sign of the cross-price effects in equation (7).

Consider an arbitrary function $\Phi(\mathbf{p})$ of type **RP** with property **C2** at a price vector $\mathbf{p}^*$ in the interior of (**dom** Φ). A general linear profit form $\Pi(\mathbf{p};\boldsymbol{\alpha})$ from equation (3) is said to have the *second-order approximation property* to Φ at $\mathbf{p}^*$ if there exists a non-negative parameter vector α^* such that the first and second derivatives of Π and Φ agree at $\mathbf{p}^*$ [i.e., $\Pi(\mathbf{p}^*;\boldsymbol{\alpha}^*) = \Phi(\mathbf{p}^*)$, $\Pi_k(\mathbf{p}^*;\boldsymbol{\alpha}^*) = \Phi_k(\mathbf{p}^*)$, and $\Pi_{kl}(\mathbf{p}^*;\boldsymbol{\alpha}^*) = \Phi_{kl}(\mathbf{p}^*)$]. The following result establishes that the general linear profit form (4) is

[4]Consider for example a technology **T** satisfying $x_2, x_3 \leqq 0$ and $x_1 \leqq [(-x_2)^{1-1/\sigma} + (-x_3)^{1-1/\sigma}]^{\mu/(1-1/\sigma)}$ for $\sigma > 0$, $\sigma \neq 1$, and $0 < \mu < 1$ (i.e., a CES production function with an elasticity of substitution σ, homogeneous of degree μ). Its restricted profit function is $\Pi = (1-\mu)\mu^{\mu/(1-\mu)} p_1^{1/(1-\mu)} (p_2^{1-\sigma} + p_3^{1-\sigma})^{-(\mu/(1-\mu))(1/(1-\sigma))}$. Π has the property GS for $\sigma \geqq 1/(1-\mu)$.

robust in the sense that locally it can mimic the net supply system of any restricted profit function with the gross substitutes property.

Lemma 1. Consider a general linear profit function Π satisfying equation (4) such that Q^{ij} has property **C2** and Q^{ij}_{rr} positive. If $\Phi(\mathbf{p})$ is any function of type **RP**, $\mathbf{p}^*$ is a vector in the interior of **dom** Φ, and Φ satisfies conditions **C2** and **GS** at $\mathbf{p}^*$, then Π has the second-order approximation property to Φ at $\mathbf{p}^*$.

Proof: From equation (7), one can choose α^*_{kl} for $k \neq l$ such that $\Pi_{kl}(\mathbf{p}^*,\boldsymbol{\alpha}^*) = \Phi_{kl}(\mathbf{p}^*)$. Then from equation (6), one can choose α^*_{kk} such that $\Pi_k(\mathbf{p}^*,\boldsymbol{\alpha}^*) = \Phi_k(\mathbf{p}^*)$. Since both Φ and Π are conical, it follows that $\Pi_{kk}(\mathbf{p}^*,\boldsymbol{\alpha}^*) = \Phi_{kk}(\mathbf{p}^*)$ and $\Pi(\mathbf{p}^*,\boldsymbol{\alpha}^*) = \Phi(\mathbf{p}^*)$. Q.E.D.

Note that the general linear profit form (4) has $N(N+1)/2$ independent parameters. This equals the number of independent conditions which must be met to obtain the second-order approximation property. In this sense, (4) is a "parameter-efficient" form among those with the approximation property.

It is clear that linear profit forms with Q-functions of more than two prices can be introduced which need not have the **GS** property; one possible form for econometric purposes will be introduced later. However, the following result shows that it is fruitless to seek a linear profit form which has the second-order approximation property to each function of type **RP** and which is itself of type **RP** over its entire domain of definition.

Lemma 2. Given any linear profit form Π in equation (3) with specified M and Q^m, there exists a function Φ of type **RP** satisfying **C2** at $\mathbf{p}^*$ in the interior of (**dom** Φ) such that Π does not have the second-order approximation property to Φ at $\mathbf{p}^*$ for $N > 2$.

Proof: Let $\mathbf{H}^*$ denote the $N-1$ matrix of derivatives $\Phi_{ij}(\mathbf{p}^*)$ for $i,j = 2,\ldots,N$, and let $\mathbf{H}^m$ denote the corresponding matrix for Q^m. For the second-order approximation property to hold, $\mathbf{H}^*$ must lie in the convex cone spanned by the $\mathbf{H}^m$. Now $\mathbf{H}^*$ can be any positive semidefinite matrix [e.g., the function

$$\Phi(\mathbf{p}) = \frac{1}{2p_1} \sum_{i,j=2}^{N} p_i p_j H^*_{ij}$$

is of type **RP** and returns this matrix]. Representing an $(N-1)$-square symmetric matrix as a point in $\mathbf{E}^{N(N-1)/2}$, the cone of positive semidefinite matrices is not polyhedral (e.g., the 2×2 submatrix

$$\begin{bmatrix} 1 & \alpha \\ \alpha & \beta \end{bmatrix}$$

is positive semidefinite on the set $\beta \geqq \alpha^2$ bounded by a parabola). Hence, $\mathbf{H}^*$ can be chosen to lie in an extreme ray of the cone which does not contain an $\mathbf{H}^m$. Q.E.D.

In view of this result, we must either restrict the class of profit functions we wish to approximate by a linear profit form, or else relax the conditions we have imposed on the linear profit form. We next give a very general result of the first type. Unfortunately, the argument is not constructive and thus does not provide a way of generating linear forms for econometric purposes.

If a function Φ of type **RP** has property **C2** and the matrix $\mathbf{H}(\mathbf{p}) = (\Phi_{ij}(\mathbf{p}))$ for $i,j = 2,\ldots,N$ is non-singular, then the dual technology at $\hat{\mathbf{x}}(\mathbf{p})$ is bounded by a surface which can be described by a twice continuously differentiable concave function $x_1 = f(\mathbf{x}_*)$ of $\mathbf{x}_* = (x_2,\ldots,x_n)$ with a non-singular Hessian matrix $f_{\mathbf{xx}}(\hat{\mathbf{x}}_*(\mathbf{p})) = [-p_1\mathbf{H}(\mathbf{p})]^{-1}$ (Chapter I.1, Theorem 26). Define an index $\rho(\mathbf{p}) =$ (minimum root of $\mathbf{H}(\mathbf{p})$)/(maximum root of $\mathbf{H}(\mathbf{p})$). Then $\rho(\mathbf{p})$ is a measure of the "relative definiteness" of the matrix $\mathbf{H}$, or equivalently of the relative curvature of the surface of the technology.[5]

Lemma 3. Given $\epsilon > 0$, there exists a linear profit form Π in equation (3) with specified M and Q^m (depending in general on ϵ) such that if Φ is any function of type **RP** satisfying **C2** at $\mathbf{p}^*$ in the interior of (**dom** Φ) and if $\rho(\mathbf{p}^*) \geqq \epsilon$ for this function, then Π has the second-order approximation property to Φ at $\mathbf{p}^*$.

Proof: As in Lemma 2, denote symmetric $(N-1)$ matrices $\mathbf{H}$ as points in $\mathbf{E}^{N(N-1)/2}$. Define the set

$$\mathbf{A}_\theta = \left\{ \mathbf{H} \in \mathbf{E}^{N(N-1)/2} \,\middle|\, \sum_{i=2}^{N} H_{ii} = 1 \text{ and } \underset{\mathbf{q}\neq 0}{\mathrm{Min}}\, (\mathbf{q}'\mathbf{H}\mathbf{q})/\mathbf{q}'\mathbf{q} \geqq \theta \right\}.$$

[5]Note that $\rho(\mathbf{p}) =$ (minimum root of $-f_{\mathbf{xx}}(\hat{\mathbf{x}}_*(\mathbf{p}))$)/(maximum root of $-f_{\mathbf{xx}}(\hat{\mathbf{x}}_*(\mathbf{p}))$), where $\mathbf{x}_* = (x_2,\ldots,x_N)$. Let $\mathbf{G}$ be a matrix with $\mathbf{G}' = \mathbf{G}^{-1}$ such that $\mathbf{G}'\mathbf{H}\mathbf{G}$ is diagonal. Define a set of "composite" commodities $\mathbf{y}_* = \mathbf{G}'\mathbf{x}_*$ and corresponding prices $\mathbf{q}_* = \mathbf{G}'\mathbf{p}_*$. Then ρ is the ratio of the smallest to the largest own price effects of the composite commodities.

Then $\mathbf{A}_\theta$ is non-empty, closed, bounded, and convex for $0 \leqq \theta < 1/(N-1)$, and for $\theta > 0$, $\mathbf{A}_\theta$ is contained in the relative interior of $\mathbf{A}_0$. Hence, for $\theta = (\text{Min}(\epsilon,.5))/(N-1)$, there exists a convex polytope with vertices $\mathbf{H}^1,\ldots,\mathbf{H}^J$ which contains $\mathbf{A}_\theta$ and is contained in $\mathbf{A}_0$. Define $Q^m = (1/2p_1)\mathbf{p}_*'\mathbf{H}^m\mathbf{p}_*$ for $m = 1,\ldots,J$, where $\mathbf{p}_* = (p_2,\ldots,p_N)$; $Q^m = p_{m-J}$ for $m = J+1,\ldots,J+N$; and $Q^m = -p_{m-J-N}$ for $m = J+N+1,\ldots,J+2N$, with $M = J+2N$. Given Φ and $\mathbf{H}(\mathbf{p}^*)$, one has

$$\epsilon \leqq \rho(\mathbf{p}^*) \leqq [\text{Min}(\mathbf{q}'\mathbf{H}(\mathbf{p}^*)\mathbf{q}/\mathbf{q}'\mathbf{q})] \Big/ \left[\sum_{i=2}^{N} H_{ii}(\mathbf{p}^*)/(N-1)\right].$$

Hence, $\mathbf{H}(\mathbf{p}^*)$ is contained in the convex cone spanned by $\mathbf{H}^1,\ldots,\mathbf{H}^J$. Choosing $(\alpha_1,\ldots,\alpha_J)$ to equate the second partials of Π and Φ, and then choosing $(\alpha_{J+1},\ldots,\alpha_M)$ to equate the first partials, yields the desired conclusion. Q.E.D.

A function Φ of type **RP** with property **C2** at a vector $\mathbf{p}$ is said to have a *dominant own price effect* with numeraire commodity 1 if

$$p_i\Phi_{ii}(\mathbf{p}) \geqq \sum_{\substack{j=2\\ j\neq i}}^{N} p_j|\Phi_{ij}(\mathbf{p})| \quad \text{for} \quad i = 2,\ldots,N.$$

From homogeneity, satisfaction of this condition requires that commodity 1 be a gross substitute for every other commodity. However, some patterns of gross complements among the remaining commodities are possible. If Φ has property **GS**, then it has the dominant own-price effect property. The next result provides a constructive proof that the class of profit functions with the dominant own-price effect property can be approximated to second order by a linear profit form:[6]

Lemma 4. Suppose good 1 is a specified numeraire commodity and $\mathbf{p}^*$ is a specified positive vector, and consider the linear profit form

$$\Pi(\mathbf{p};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \sum_{i=1}^{N} \alpha_i p_i + \frac{1}{2p_1}\sum_{i=2}^{N}\sum_{j=2}^{N}\left[\beta_{ij}\left(\frac{p_i}{p_i^*}+\frac{p_j}{p_j^*}\right)^2 + \gamma_{ij}\left(\frac{p_i}{p_i^*}-\frac{p_j}{p_j^*}\right)^2\right], \tag{8}$$

with β_{ij} and γ_{ij} non-negative and symmetric in $ij, \gamma_{ii} = 0$, and $\beta_{ij}\cdot\gamma_{ij} =$

[6]The restricted profit function given in footnote 4 has the dominant own-price effect property if $\sigma \geqq 1/2(1-\mu)$, or if $\sigma < 1/2(1-\mu)$ and $(1-2\sigma(1-\mu))^{-1} \geqq (p_2/p_3)^{1-\sigma} \geqq 1-2\sigma(1-\mu)$. In particular, this property holds for all positive prices in the limiting case $\sigma = 1$.

0. If Φ is any function of type **RP** satisfying **C2** and the dominant own price effect property at $\mathbf{p}^*$ in the interior of $(\mathbf{dom}\ \Phi)$, then (8) has the second-order approximation property to Φ at $\mathbf{p}^*$.

Proof: Differentiating (8), we obtain

$$\Pi_{kj} = 2(\beta_{kl} - \gamma_{kl})/p_1 p_k^* p_l^* \quad \text{for} \quad k \neq l, \tag{9}$$

$$\Pi_{kk} = 2\beta_{kk}/p_1 p_k^{*2} + 2 \sum_{j=2}^{N} (\beta_{kj} + \gamma_{kj})/p_1 p_k^{*2}. \tag{10}$$

Condition (9) with $\Pi_{kl} = \Phi_{kl}(\mathbf{p}^*)$ and $\beta_{kl} \cdot \gamma_{kl} = 0$ determines β_{kl}, γ_{kl} for $k \neq l$. Substituting these values in (10) yields

$$\begin{aligned} \beta_{kk}/p_1^* p_k^* &= p_k^* \Phi_{kk}(\mathbf{p}^*) - \frac{2}{p_1^* p_k^*} \sum_{j=2}^{N} (\beta_{kj} + \gamma_{kj}) \\ &= p_k^* \Phi_{kk}(\mathbf{p}^*) - \sum_{j=2}^{N} p_j^* |\Phi_{kj}(\mathbf{p}^*)| \geqq 0, \end{aligned}$$

by the dominant own price effect property. Choose the α_i to equate the first partials of Π and $\Phi(\mathbf{p}^*)$. This establishes the desired result. Q.E.D.

By weakening the requirement that a general linear profit form be finite for all positive prices, one can obtain the second-order approximation property to a broad class of restricted profit functions, as shown in the following result. This is in effect an unrestricted *local* approximation theorem.

Lemma 5. Consider the linear profit form (4) with specified twice continuously differentiable Q^{ij} having Q^{ij}_{rr} non-zero. Suppose Φ is of type **RP**, has property **C2** at $\mathbf{p}^*$ in the interior of $\mathbf{dom}\ \Phi$, and has $\mathbf{H}(\mathbf{p}) = (\Phi_{ij}(\mathbf{p}))$, $i,j = 2,\ldots,N$ non-singular at $\mathbf{p}^*$. Then there exists a parameter vector $\boldsymbol{\alpha}^*$ in (4), not necessarily non-negative, and a closed cone $\mathbf{P}^*$ containing $\mathbf{p}^*$ in its interior such that the function $\Pi^*(\mathbf{p})$, defined to equal $+\infty$ for $\mathbf{p} \notin \mathbf{P}^*$ and to equal (4) for $\boldsymbol{\alpha}^*$ and $\mathbf{p} \in \mathbf{P}^*$, is of type **RP** and has the second-order approximation property to Φ at $\mathbf{p}^*$.

Proof: From the proof of Lemma 1, $\boldsymbol{\alpha}^*$ can be chosen so that $\Pi(\mathbf{p};\boldsymbol{\alpha}^*)$ has the second-order approximation property to Φ at $\mathbf{p}^*$. Since $\mathbf{H}(\mathbf{p}^*)$ is positive definite, it follows from continuity that $\Pi(\mathbf{p};\boldsymbol{\alpha}^*)$ is convex for $\mathbf{p}$ in a convex neighborhood of $\mathbf{p}^*$; let $\mathbf{P}^*$ be the smallest closed convex cone containing this neighborhood. It is then immediate that $\Pi^*(\mathbf{p})$ is of type **RP**. Q.E.D.

Note that this result does not require the Q-functions to be convex. Thus for the purposes of this type of approximation, one could take equation (4) to have a wide variety of functional forms, such as (5) with unrestricted signs, or the Christensen–Jorgenson–Lau (1973) "translog" function. The difficulty with using Lemma 5 as a justification for choice of simple functional forms for econometric analysis without regard for global properties of the Q-functions is that the domain $\mathbf{P}^*$ cannot be determined *a priori* and may not include all observations. Use of a fitted equation (4) without restriction of domain may be inconsistent even locally with competitive profit maximization. An ex post consistency check for this inclusion is typically highly non-linear and computationally forbidding. However, Appendix A.4 by Lau has established feasible methods of testing the convexity of a function at each data point.

4. The Dual Technology of the General Linear Profit Function

The lemmas above giving second-order approximation properties of the general linear profit function to an arbitrary function of type **RP** can be interpreted dually as establishing that an arbitrary convex technology can be mimicked locally by the dual technology of the linear profit form. Beyond this conclusion, it is useful to establish some of the global properties of the dual linear technology.

We noted earlier that when the linear profit form is linear in prices, it is dual to a Leontief fixed coefficients technology. All the forms we consider yield this technology as a special case. More generally, we can from Theorem 2 express the dual technology of the general linear profit form (3) as a sum (see Chapter I.1, Table 5),

$$\mathbf{T} = \sum_{m=1}^{M} \alpha_m \mathbf{T}^m, \tag{11}$$

where

$$\mathbf{T}^m = \{\mathbf{x} | \mathbf{p} \cdot \mathbf{x} \leqq Q^m(\mathbf{p}) \text{ for all } \mathbf{p}\}. \tag{12}$$

When the Q-functions are simple forms, the sets $\mathbf{T}^m$ can often be characterized explicitly. The last column of Table 1 lists the dual technologies corresponding to a variety of two-price Q-functions. The three-price functional form (8) has the Q-function $(p_i/p_i^* + p_j/p_j^*)^2/2p_1$ dual to a technology $\mathbf{T}_+^{ij}$ with $x_k \leqq 0$ for $k \neq i,j$, and $\text{Max}\,[p_i^* x_i, p_j^* x_j] \leqq \sqrt{(-2x_1)}$; and the Q-function $(p_i/p_i^* - p_j/p_j^*)^2/2p_1$ dual to a technology T_-^{ij} with $x_k \leqq 0$ for $k \neq i,j$, $p_i^* x_i + p_j^* x_j \leqq 0$, and $\text{Max}[p_i^* x_i, p_j^* x_j] \leqq \sqrt{(-2x_1)}$. The structure (11) of the dual technology has a direct economic interpretation of non-jointness of the component technologies $\mathbf{T}^m$, implying that one can "decentralize" the optimization decisions in these components. In Chapter II.4, several examples are given in which this structure arises naturally for a multiple production unit firm.

The technological structure of equations (11) and (12) can also be characterized by "transformation" or gauge functions for the technologies. Let $\mathbf{e}$ denote a vector of ones, and for functions Q^m in the general linear profit form (3), define $\mathbf{x}^{*^m} = Q_{\mathbf{p}}^m(\mathbf{e}) - \mathbf{e}$, where $Q_{\mathbf{p}}^m$ is the vector of partial derivatives of Q^m, or more generally any optimal net supply vector for Q^m at the price vector $\mathbf{e}$. Then $\mathbf{x}^{*^m}$ is an interior point of $\mathbf{T}^m$. Define

$$F^m(\mathbf{x}) = \text{Inf}\{\lambda > 0 | \mathbf{p}\cdot\mathbf{x} \leqq \lambda(Q^m(\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^{*^m}) \text{ for all } \mathbf{p}\}. \tag{13}$$

Then $\mathbf{x} \in \mathbf{T}^m$ if and only if $F^m(\mathbf{x} - \mathbf{x}^{*m}) \leqq 1$ (Chapter I.1, Theorem 24). Assume $\boldsymbol{\alpha}$ strictly positive and define

$$F(\mathbf{x}) = \text{Inf}\left\{\underset{m}{\text{Max}}\, \frac{1}{\alpha_m} F^m(\mathbf{x}^m) \middle| \sum_{m=1}^{M} \alpha_m \mathbf{x}^m = \mathbf{x}\right\}. \tag{14}$$

Then $\mathbf{x} \in \mathbf{T}$ if and only if $F(\mathbf{x} - \sum_{m=1}^M \alpha_m \mathbf{x}^{*^m}) \leqq 1$ (Chapter I.1, Corollary 29). In the special case where the Q^m functions are separable, depending on disjoint subsets of commodity prices, the transformation function (14) has a corresponding separable structure. Linear profit forms chosen for econometric purposes usually contain this structure as a special case that can be tested as a linear hypothesis.

For the special two-price linear form (5) with $Q^{ij}(\mathbf{p}) = -(p_i p_j)^{1/2}$, an ingenious argument of Diewert (1971) provides an analytic characterization of the dual technology. Define $x_i^* = \max(0, -\alpha_{ii})$, let $\mathbf{A}$ denote the matrix of parameters α_{ij}, and for a vector $\mathbf{x}$ let $\hat{\mathbf{x}}$ denote a diagonal matrix constructed from the components of $\mathbf{x}$. For $\mathbf{x} \ll \mathbf{x}^*$, define $f(\mathbf{x})$ to be the reciprocal of the Frobenius root of the non-negative matrix

$$(\widehat{\mathbf{x}^* - \mathbf{x}})^{-1/2}(\hat{\mathbf{x}}^* + \mathbf{A})(\widehat{\mathbf{x}^* - \mathbf{x}})^{-1/2}.$$

Then f is a concave function, and $\mathbf{x} \in \mathbf{T}$ if and only if $f(\mathbf{x}) \geqq 1$.[7] Unfortunately this type construction does not seem to carry over to other two-price functional forms.

Finally, in the case of two commodities, the linear profit form (4) has a simple geometric interpretation: The function $Q^{12}(p_2/p_1)$ is dual to a "one input–one output" production function or an "isoquant" in the negative quadrant. The technology (11) is defined by shifting this surface by a scale factor and then shifting the axes. In Diewert's form (5) the surface is a translated rectangular hyperbola.

5. Applications of the Linear Profit Function

Our interest in the general linear profit function is based on its linear-in-parameters form, which allows estimation of the net supply system by linear regression methods. We now suggest several ways in which this structure can be exploited. The first comment concerns constant returns technologies.

(1) Use of the derivative property to obtain the net supply system under the assumption that the restricted profit function is differentiable with a Hessian of full rank implies that the dual technology is strictly convex. This condition is inconsistent with the assumption of a constant returns technology, and more fundamentally the specification of a constant returns technology and competitive profit maximization is insufficient to determine a net supply function describing the behavior of the firm. An obvious and reasonable way to obtain a definite net supply is to assume that the firm at each point in time treats as fixed some durable inputs which are essential to production and maximizes profit in variable goods, and then adjusts durable inputs over time subject to, say, equity constraints. The "per durable input unit" technology can then be strictly convex, and the formulae (3) and (6) specify a "per unit" net

[7]The argument is based on equation (2), which implies $\mathbf{x} \in \mathbf{T}$ if and only if

$$f(\mathbf{x})^{-1} = \sup\left\{\left(\sum_{i,j} p_i^{1/2} p_j^{1/2} a_{ij} + \mathbf{p}\cdot\mathbf{x}^*\right) \Big/ \sum_i p_i(x_i^* - x_i)\right\} \leqq 1.$$

Defining $q_i = p_i^{1/2}(x_i^* - x_i)^{-1/2}$, and $\hat{\mathbf{x}} = \operatorname{diag}(x_i)$, this formula becomes

$$f(\mathbf{x})^{-1} = \sup\{\mathbf{q}'(\widehat{\mathbf{x}^* - \mathbf{x}})^{-1/2}(\hat{\mathbf{x}}^* + \mathbf{A})(\widehat{\mathbf{x}^* - \mathbf{x}})^{-1/2}\mathbf{q}/\mathbf{q}'\mathbf{q}\},$$

and the result follows from the theory of non-negative matrices.

supply system, with the value of Π interpreted as the implicit rate of return on the durable input.

(2) Our second comment concerns aggregation over firms $f = 1,\ldots,F$ which have a common technology characterized by a parameter vector $\boldsymbol{\alpha}$, but face differing price vectors $\mathbf{p}_f$, and $\Pi(\mathbf{p}_f;\boldsymbol{\alpha})$ denotes the restricted profit function of firm f. Then aggregate profit equals $\Sigma_{f=1}^{F} \Pi(\mathbf{p}_f;\boldsymbol{\alpha})$, and aggregate net supply is given by the corresponding sum of price derivatives. In the linear-in-parameter form (3) for the profit function, this aggregation "carries past" the parameters, preserving the linear structure,

$$\sum_{f=1}^{F} \Pi(\mathbf{p}_f;\boldsymbol{\alpha}) = \sum_{m=1}^{M} \alpha_m \left[\sum_{f=1}^{F} Q^m(\mathbf{p}_f)\right]. \tag{15}$$

The parameters of this problem could then be estimated from aggregate net supply data and disaggregated price data. The system has an obvious application when detailed price data is available but disclosure rules prevent the release of detailed quantity data. In practice, something less than completely disaggregate price data may be sufficient to compute the expressions $[\Sigma_{f=1}^{F} Q^m(\mathbf{p}_f)]$. For example, with the Diewert specification (5) of the linear profit form involving terms $p_i Q^{ij} = -(p_i p_j)^{1/2}$, if the mean μ_i of p_{fi} and covariance σ_{ij} of $p_{fi} p_{fj}$ across firms are reported, and if one can make the maintained hypothesis that $\mathbf{p}_f$ is multivariate log-normally distributed and the number of firms in the aggregate is large, then one has

$$\frac{1}{F} \sum_{f=1}^{F} [-(p_{fi} p_{fj})^{1/2}] \cong -(\mu_i \mu_j)^{1/2} \frac{(1 + \sigma_{ij}/\mu_i \mu_j)^{1/4}}{(1 + \sigma_{ii}/\mu_i^2)^{1/8} (1 + \sigma_{jj}/\mu_j^2)^{1/8}}. \tag{16}$$

Then a series of observations on aggregate net supply and means and covariances of prices within each aggregate observation would be sufficient to estimate the model.

Another interpretation of the system (15) and (16) can be given for a single firm facing uncertainty, with f denoting the state of nature. The corresponding net supply system expresses expected net supplies as linear-in-parameters functions of the means and covariances of prices.

A second class of aggregation problems occurs when firms $f = 1,\ldots,F$ face common commodity prices, but have different technologies with non-measured "local" factors. If the restricted profit function of each firm is of the general linear form, with the parameter vector $\boldsymbol{\alpha}_f$ differing across firms, but the Q-functions common to all firms, then aggregate net supplies can be interpreted as coming from a "representative" technology of the same form with a parameter vector $\boldsymbol{\alpha} = (1/F)\sum_{f=1}^{F} \boldsymbol{\alpha}_f$. [See Klein (1952–53).]

(3) Our third comment concerns tests of restrictions on the technology. In a number of cases, these can be formulated as linear restrictions on the parameter vector $\boldsymbol{\alpha}$, and thus tested using standard linear statistical theory. For example, suppose goods 1 and 2 are outputs, the remaining commodities are inputs and we wish to test the non-jointness of production of the two commodities. This hypothesis implies that the net supply of good 1 cannot be affected by the price of good 2, i.e., $\alpha_{12} = 0$ in the two-price linear profit form (4).

(4) Our fourth comment concerns the introduction of exogenous variables from the technology into the restricted profit function when it is assumed to have the linear profit form. Important cases include cost minimization for fixed output and profit maximization in a subset of commodities with the remaining commodity levels fixed. These variables will typically enter in a non-linear way (except in the case of constant returns). However, one can introduce a linear-in-parameter form jointly over the variable commodity prices and exogenous variables, and establish second-order approximation properties for this form in both sets of variables. For example, if the profit function has arguments $\mathbf{p} = (p_1,\dots,p_N)$ and $\mathbf{z} = (z_{N+1},\dots,z_L)$, and the underlying technology is convex in $(\mathbf{x},\mathbf{z})$, then the profit function is concave in $\mathbf{z}$ and one might consider a linear form,

$$\Pi(\mathbf{p},\mathbf{z}) = \sum_{i,j=1}^{N} \alpha_{ij}(-(p_i p_j)^{1/2}) + \sum_{i=1}^{N} \sum_{j=N+1}^{L} \beta_{ij} p_j z_j + p_1 \sum_{\substack{i,j=N+1 \\ i \neq j}}^{L} \gamma_{ij}(z_i z_j)^{1/2} + p_1 \sum_{i=N+1}^{N} \delta_i(-z_i^2), \tag{17}$$

in $L(L+1)/2 + (L-N)$ parameters with α_{ij} and γ_{ij} non-negative and symmetric, δ_i non-negative. This form has the second-order approximation to any function $\Phi(\mathbf{p},\mathbf{z})$ which is of type **RP** in $\mathbf{p}$, is concave in $\mathbf{z}$, is twice continuously differentiable jointly in $(\mathbf{p},\mathbf{z})$, has the property **GS** in $\mathbf{p}$, and has a dual **GS** property in $\mathbf{z}$ (i.e., the marginal product $\partial\Pi/\partial z_i$ is non-decreasing in all other z_j).

Our last comment concerns the construction of "nested" functional forms for the restricted profit function which can be interpreted as arising from a two-stage decision process (ex ante and ex post) of the firm. Suppose the linear profit form (3) summarizes the result of ex post optimization, with the α_m which are fixed ex post being the ex ante decision variables. Suppose these ex ante decision variables are described parametrically by a second linear profit form,

$$\psi(q_1,...,q_m) = \sum_{l=1}^{L} \beta_l R^l(q_1,...,q_m),$$

with β non-negative, R^l non-decreasing in $\mathbf{q}$, and

$$\alpha_m = \partial\psi/\partial q_m. \tag{18}$$

Then, the optimal ex ante profit maximum is found to equal $\psi(Q^i(\mathbf{p}),...,Q^m(\mathbf{p})) \equiv \Phi(\mathbf{p})$, which is of type **RP**, and the optimal net supply vector resulting from the two-stage optimization is found to satisfy

$$\hat{x}_i(\mathbf{p}) = \sum_{l=1}^{L} \beta_l \sum_{m=1}^{M} R^l_m(Q^1(\mathbf{p}),...,Q^M(\mathbf{p}))Q^m_i(\mathbf{p}), \tag{19}$$

and

$$\hat{\alpha}_m(\mathbf{p}) = \sum_{l=1}^{L} \beta_l R^l_m(Q^1(\mathbf{p}),...,Q^M(\mathbf{p})). \tag{20}$$

This structure is linear in the underlying production parameters β_l. A detailed discussion of ex ante–ex post production structures and their estimation by nested linear profit forms is given in Chapter II.4.