Problem Set 1

Due: Tuesday, September 11, 2007

1. Let $\{X_t\}$ be a continuous-state discrete-time Markov process taking values on the unit interval. Suppose the conditional distribution of X_t given $X_{t-1} = y$ is uniform on the interval $(1-y, 1)$. Show that the process is stationary if the marginal density of X_t is $2x$ for x in the unit interval.

2. Let $\{y_t\}$ be the MA(1) process generated as $y_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where $|\theta| < 1$ and $\{\varepsilon_t\}$ is Gaussian white noise with zero mean and unit variance. Define

$$z_t = \begin{cases} 1 & \text{if } y_t \geq 0 \\ 0 & \text{if } y_t < 0 \end{cases}$$

   Show that $\{z_t\}$ is stationary. Derive its autocorrelation function. What is the maximum value that the first autocorrelation coefficient can take? [Hint: Use the fact that, if X and Y are bivariate normal with zero means, unit variances and covariance r, the probability that both X and Y are negative is given by $g(r) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1} r$.]

3. Compute the autocorrelation function and the partial autocorrelation function of the AR(3) process

$$y_t = 0.6y_{t-3} + \varepsilon_t$$

   where ε_t is white noise with mean zero and variance σ^2.

4. Let $\{y_t\}$ be a stationary AR(p) process generated as $A_p(L)y_t = \varepsilon_t$, where $\{\varepsilon_t\}$ is a mean zero white-noise sequence with unit variance. Let $\{\rho_k\}$ be its autocorrelation sequence. Suppose the process has the moving average representation $y_t = C(L)\varepsilon_t$ where $C(L) = 1 + c_1L + c_2L^2 + \cdots$. Show that $\mathrm{E}y_t\varepsilon_{t-s} = c_s$ and hence the moving average coefficients satisfy $A(L)c_s = 0$ when $s > 0$. Show also that$A(L)\rho_s = 0$ when $s > 0$. Does this imply $c_s = \rho_s$ for all s?

5. Consider the stationary ARMA(1,1) process generated by

$$y_t = \alpha y_{t-1} + \varepsilon_t + \beta\varepsilon_{t-1}$$

   where the ε's are white noise with mean zero and variance one. Assuming that both α and β are positive and less than one, show that the autocovariances $\gamma_r = \mathrm{E}y_t y_{t-r}$ are monotonically decreasing in $|r|$.

6. Suppose $\{x_t\}$ and $\{y_t\}$ are stationary AR(1) processes with representations

$$x_t = \alpha x_{t-1} + \varepsilon_t \qquad\qquad y_t = -\alpha y_{t-1} + \eta_t$$

   where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are independent white noise processes, each with mean zero and unit variance. Define $z_t = x_t + y_t$.

   (a) Show that z_t is a stationary AR(2) process.

   (b) Find the best linear predictor of z_t given the infinite past history $z_{t-1}, z_{t-2}, ...$ Find the variance of the prediction error.

   (c) Find the best linear predictor of z_t given the infinite past history $x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, ...$ Find the variance of the prediction error.