Econ 201a Part II
Fall 2005

Problem Set 1 – Solutions

Problem 1.

(a) (not for submission) No. The game below has a unique PSNE (C, c). Equilibrium dynamics involves a cycle (A, a) – (A, b) – (B, b) – (B, a) – (A, a) - …, which never converges to (C, c).

	a	b	C
A	2, -2	-2, 2	1, 1
B	-2, 2	2, -2	1, 1
C	1, 1	1, 1	2, 2

(b) Yes. Suppose (X, Y) is a unique pair of strategies that survives iterative elimination of strictly dominated strategies. The proof that the players converge to (X, Y) is by contradiction. If the dynamics does not converge to (X, Y), then assuming that best responses are unique there is a cycle

$(a_1, b_1) – (a_1, b_2) – (a_3, b_2) – (a_3, b_4)$ - … $(a_{2n-1}, b_{2n}) – (a_1, b_{2n})$ - …

with $b_{2n} = b_1$. In the process of iterative deletion of strictly dominated strategies, one of the actions involved in this cycle must be eliminated first. Without loss of generality it is a_1. Then a_1 is eliminated before b_{2n}. This leads to a contradiction because a_1 is a best response to b_{2n}, so we cannot eliminate a_1 before b_{2n}.

Remark: It is possible that the best responses are not unique. Then there may be no cycle. However, we can still find identical pairs of actions $(a_1, b_1) = (a_{2n+1}, b_{2n})$ along the path of play, because each player has only finitely many actions. In the path between (a_1, b_1) and (a_{2n+1}, b_{2n}) one of the actions must be eliminated first, which leads to a contradiction because each action in this path must be a best response to another action in the same path.

(c) No. The counterexample from part (a) works here also. (½ A + ½ B, ½ a + ½ b) is not a mixed strategy Nash equilibrium because each player has a profitable deviation to c.

Problem 2.

Answer: players can cooperate for n-2 periods, but not more.

Let us show that there is a SPE in which the equilibrium path is

$$\underbrace{(C,C)-(C,C)-...-(C,C)}_{n-2}-(D,C)-(C,D)-(B,B) \qquad (*)$$

and players switch to punishment path *(D, D) - … (D, D)-(F,F)* if player 1 deviates and *(D, D) - … (D, D)-(B, B)* if player 2 deviates. Observe that the punishment paths are SPE in all subgames. Let us verify that neither player will want to deviate from the path (*). If player 1 deviates in period n, he gains 1 in period n and the last period payoffs become (1, 2) instead of (2, 1). Therefore, this deviation is not profitable. If player 2 deviates in period $n-1$ he gains 1 immediately, but the payoffs in the following subgame change from (-1, 2) – (1, 2) to (0, 0) – (1, 2). This deviation is not profitable. For periods $k \leq n-1$, note that a deviation by player 1 causes payoffs in the last 3 periods to change from

(2, -1) – (-1, 2) – (2, 1)   to   (0, 0) – (0, 0) – (1, 2)

Clearly, such a deviation is not profitable because it has a loss of at least 2 and gain of only 1. Similarly, we can show that a deviation by player 2 has a loss of at least 1, so it is not profitable either.

To see that cooperation in period n is impossible, note that the payoffs in the last period are (1, 2) or (2, 1). In either case, one of the two players gets a payoff of 1, and we cannot provide incentives to cooperate to both players simultaneously. However, we can sustain (C, D) and (D, C) because we *can* provide incentives to one of the players.

To see that cooperation in period n-1 is impossible, note that the equilibrium path in the last two periods has payoffs (0, 0) – (1, 2), (0, 0) – (2, 1), (2, -1) – (1, 2) or (-1, 2) – (2, 1). Again, in each case one of the two players gets a payoff of 1, so we cannot provide incentives to cooperate to both players simultaneously.

Problem 3.

(x, 100 - x) is a NE for any $x \in [0, 100]$. We also have equilibria of the form (x_1, x_2) where $x_1, x_2 \geq 100$.

Problem 4.

A strategy includes one action from each information set. Thus, the number of strategies is just $\prod_{n=1}^{N} M_n$.

Problem 5.

(a) For player 1, E is strictly dominated by A. For player 2, a is strictly dominated by any mixture of b and e, or d and e.

(b) Iterative deletion leaves (A, C, D) for player 1 and (b, c, e) for player 2, so the game is not dominance solvable.

(c) We only need to check the strategies that survive iterative deletion.
(C, c) is the unique PSNE.

(d) We don't need to consider strategies eliminated by iterative deletion of strictly dominated strategies. Thus, we consider the reduced 3 x 3 game:

	b	c	e
A	4, 2	1, 0	2, 4
C	0, 0	3, 3	0, 1
D	2, 4	0, 0	4, 2

First note that the unique PSNE (C, c) is a special case of a MSE.

Second, note that neither player has two pure best responses to any opponent's pure strategy, so there will be no pure/mixed equilibria.

We now look for equilibria where each player mixes over all 3 strategies:

Player 1		Player 2	
A	λ_1	b	μ_1
C	λ_2	c	μ_2
D	$(1-\lambda_1-\lambda_2)$	e	$(1-\mu_1-\mu_2)$

We pin down μ_1 and μ_2 by requiring that 1(!) be indifferent between A, C and D.

$$\begin{aligned} U_1(A) &= 4\mu_1 + \mu_2 + 2(1-\mu_1-\mu_2) \\ U_1(C) &= 3\mu_2 \\ U_1(D) &= 2\mu_1 + 4(1-\mu_1-\mu_2). \end{aligned}$$

Equating these three yields μ_1=1/11, μ_2 = 6/11.

Analogously, requiring 2's indifference yields λ_1 = 1/11 and λ_2 = 6/11.

Thus, we have the MSE (1/11, 6/11, 4/11), (1/11, 6/11, 4/11).

Now we look for mixed strategies in which at least one player plays only 2 strategies. Note the following (using dominance against the strategies in question):

If 1 plays A and C only, 2 will only play c or e;
if 1 plays C and D only, 2 will only play b or c;
if 1 plays A and D only, 2 will only play b or e.

Similarly:

If 2 plays b and c only, 1 will only play A or C;
if 2 plays c and e only, 1 will only play C or D;
if 2 plays e and b only, 1 will only play A or D.

From this, we can rule out any 2×3 MSE. We also see that the only potential 2×2 MSE are of the form $\lambda A + (1-\lambda)D, \mu b + (1-\mu)e$.

Using the same technique as in the 3×3 case, we find $\mu = \lambda = \frac{1}{2}$.

Thus, we have a third and final MSE: $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})$.

Bonus Problem: The monopolist's payoff function is

$$\sum_{t=1}^{T} p_t x_t$$

where x refers to the quantity of sales.

Before finding the optimal path of prices we should recall that in period k-1 customer v_k must be indifferent between buying in period k-1 or k. The customer indifference condition is

$$(n+1-(k-1))\, v_k - p_{k-1} = (n+1-k)\, v_k - p_k \;\Rightarrow$$
$$v_k = p_{k-1} - p_k$$

To solve this problem we proceed by backward induction. We will transform the problem by incorporating the above equation in the monopolist's payoff function and making it solely a function of the sequence of v.

At period t-1 we take the value of v_{t-1} as given and solve for the optimal value of v_t given v_{t-1}. This leads to a Dynamic Programming problem.

Let us conjecture and verify inductively that that the monopolist's profit in period k is given by $\pi_k = \alpha_k v_k^2$, and the price is given by $\beta_k v_k$ for some sequence β_k and $\alpha_k = \beta_k/2$. We will derive a recursive equation for β_k. From class we know that the boundary values are $\beta_n = ½$ and $\alpha_n = ¼$.

Assuming that form of profit and price in period k, let us derive it for period k-1. In period k-1 customer v_k must be indifferent between buying in period k-1 or k. That customer's indifference condition is

$$(n+1-(k-1))\, v_k - p_{k-1} = (n+1-k)\, v_k - \beta_k v_k \;\Rightarrow$$
$$(1+\beta_k)\, v_k = p_{k-1}.$$

In period k-1 the monopolist will choose price p_{k-1} to maximize profit in period k-1 and future profit

$$(v_{k-1} - v_k)p_{k-1} + \beta_k v_k^2/2 = (v_{k-1} - v_k)(1+\beta_k)v_k + \beta_k v_k^2 / 2 .$$

Taking F.O.C. with respect to v_k we find

$$(1+\beta_k)v_{k-1} - 2(1+\beta_k)v_k + \beta_k v_k \Rightarrow v_k = \frac{(1+\beta_k)v_{k-1}}{2+\beta_k}$$

Then the principal's profit in period k-1 is

$$\left(v_{k-1} - \frac{(1+\beta_k)v_{k-1}}{2+\beta_k}\right)(1+\beta_k)\frac{(1+\beta_k)v_{k-1}}{2+\beta_k} + \beta_k\left(\frac{(1+\beta_k)v_{k-1}}{2+\beta_k}\right)^2 / 2 = \frac{(1+\beta_k)^2}{4+2\beta_k}{v_{k-1}}^2,$$

Therefore, $\alpha_{k-1} = \dfrac{(1+\beta_k)^2}{4+2\beta_k}$. Also,

$$p_{k-1} = (1+\beta_k)\, v_k = \frac{(1+\beta_k)^2 v_{k-1}}{2+\beta_k} \Rightarrow \beta_{k-1} = \frac{(1+\beta_k)^2}{2+\beta_k} = 2\alpha_{k-1}.$$

This proves inductively that $\beta_k = 2\alpha_k$ for all periods.

<u>Answer:</u> The recursive formulas are $v_{k-1} = \dfrac{(2+\beta_k)v_k}{1+\beta_k}$, $p_k = \beta_k v_k$, where β are defined recursively by $\beta_n = ½$, $\beta_{k-1} = \beta_k + \dfrac{1}{2+\beta_k}$.

Let us show that the principal gets less than monopoly profit. Note that the monopoly profit with commitment is $n/4$, which is obtained by setting the price equal to $n/2$ for all periods. We would like to show that $\pi_1 = \beta_1/2 < n/4$. This follows because $\beta_n = 1/2$ and

$$\beta_{k-1} = \beta_k + \frac{1}{2+\beta_k} < \beta_k + \frac{1}{2}.$$

Alternatively you could consider the case where T=∞. Here the problem is not well defined since the profit function is unbounded. Let's introduce discounting by discount factor δ. Here in accord with the Coase-conjecture, as δ→1 the monopolist's profit tends to zero.

Equivalently to this solution one can also formally write down the Dynamic Programming problem and solve that by backward induction.

For further reference, see:

Sobel and Takahashi (1983) A multi-stage model of bargaining *Review of Economic Studies* 50: 411-426
Chapter 10 of Fudenberg & Tirole Game Theory
Coase-conjecture on the web