Econ 201A. Part II.

Solutions to Problem Set 2

Problem 1. Consider a prisoner's dilemma repeated n times

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

→ . . .

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

→

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

A grim-trigger strategy in a Prisoner's dilemma is a strategy that plays C as long as the opponent cooperates, and plays D for all subsequent periods after the opponent defects. For example if the opponent plays CCDCDDD, then a grim-trigger strategy will play CCCDDDD.

(a) If two grim-trigger strategies are playing against each other, what path of play will happen?

The outcome will be (C, C) in every period.

(b) If a fully rational player has a grim-trigger opponent, how will he play? In other words, what is a best response to a grim-trigger strategy?

A fully rational player will cooperate for n-1 periods, defect in the last period and gain a total payoff of n+1. To see that this is the maximal payoff that a rational player can achieve, note that there is at most one period in which the rational player defects and gets a payoff of 2 (while the opponent cooperates); in all other periods the rational player gets a payoff of at most 1.

For the next three parts, suppose that player 2 can be one of two types: *normal* with probability 1-p or *behavioral* with probability p. A normal type fully rational, but a behavioral type plays the grim-trigger strategy. Player 1 is normal. Of course, the probability that player 1 assigns to player 2 being behavioral can change depending on the path of play. For parts (c)-(e), assume that $n = 2$.

(c) How will the normal type of player 2 play?

In the period 2, the normal player 2 will always defect (there is no incentive to cooperate, regardless of player 1's current or previous actions.) Likewise in period 2, regardless of what type s/he is facing, player 1 will always defect as well. Therefore, in period 1, a normal type of player 2 has no incentive to affect player 1's beliefs about his type (i.e. to cooperate, as a grim-trigger player would) and will therefore defect in period 1 as well.

(d) If p =3/4, what happens in a PBE?

(e) Describe what can happen in a PBE for different values of p.

Answer to (d) and (e): As established above, player 1 will defect in period 2 no matter which type of player he faces. In period 1, if he plays cooperate, he gets an expected total payoff of (1-p) [-1 + 0] + p [1 + 2] = 4p-1 against the normal and behavioral type, respectively. If he plays defect, he gets an expected total payoff of (1-p) [0 + 0] + p [2 + 0] = 2p. Therefore, player 1's best first-period action is to cooperate if p ≥ ½ , and defect if ½ ≥ p; the "normal" player 2 will defect in period 1, and in the second period both players defect no matter what.

Part (f) is a very hard bonus part.

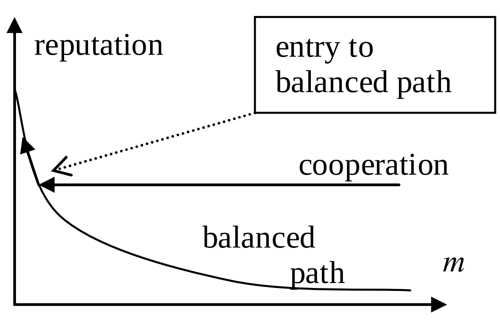
(f) Assume that p and n are arbitrary. Describe a PBE.

Let us outline the PBE and justify its main components. Some of the details are omitted. Denote by p_m (reputation) the probability that player 1 assigns to 2 being behavioral when there are m periods remaining. The main ingredient of the PPE below is that there is a "balanced path" (set of values of p_m and m, with reputation increasing as m decreases) on which the players are indifferent between cooperating and defecting.

$$p_m = \begin{cases} 2^{-m/2} & \text{if } m \text{ is even} \\ 2^{-(m-1)/2} & \text{if } m \text{ is odd} \end{cases}$$

Roughly speaking, if player 2's reputation is above the balanced path, then both players will cooperate. As a result, player 2's reputation stays the same and m decreases. Eventually, players will have to enter the balanced path when m becomes sufficiently low. If player 2's reputation is below the balanced path, then the normal type of player 2 must defect with sufficiently high probability so his reputation reaches the balanced path.

The PPE is quite complicated but the most important observation about it is this: for any positive initial level of reputation, players will cooperate in all but finitely many periods in the end of the game. The following figure schematically shows the equilibrium dynamics:



The balanced path: *If $p_m = 1/2^{m/2}$ when m is even or $p_m = 1/2^{(m-1)/2}$ when is odd, let us show that there is a PBE in which the following happens along the path of play:*

If cooperation has not broken down:

A. If an odd number 2k+1 of periods are remaining, player 1 has belief $p_{2k+1}=1/2^k$. Player 2 cooperates and player 1 mixes with probability ½. Player 1's belief does not change after this period.
B. If there is an even number 2k of period remaining, then player 1 cooperates and player 2 cooperates and defects with probability ½ each. For player 2, this means that the grim-trigger player always cooperates and normal player 2 mixes with such probabilities that the total probability that player 2 (both normal and behavioral) cooperates is ½. Posterior probability that player 2 is behavioral rises by a factor of 2.
C. If one period remains, player 1 defects and player 2, a grim-trigger player for sure, cooperates.

If one of the players previously defected, then cooperation breaks down: both players defect in all subsequent periods.

Let us show that if cooperation has not broken down, then players are indifferent between cooperating and defecting at every stage until the last stage for player 1 and the second to last stage for player 2. This follows because each player is always indifferent between defecting in the current stage or in the next stage until the "end of the game," as the following argument shows:

Player 2: If he defects currently, he gets a payoff of 0 in all subsequent periods. If he cooperates currently and defects in the next period then he worsens his current-period payoff by 1, but improves his payoff in the next period by 1 (because player 1 will cooperate or defect with probability ½: because he mixes in one of the two periods.)

Player 1: If he defects when player 2 is mixing, he gets an expected payoff of 1 currently. If he cooperates in the current period and defects in the next, he gets an expected payoff of 0 currently (1 if player 2 cooperates and -1 if player 2 defects). With probability ½ player 2 happens to cooperate, and player 1 gets a payoff of 2 in the next period. With probability ½ player 2 defects, and player 1 gets a payoff of 0.

If player 1 defects when player 2 is cooperating for sure, he gets a payoff of 2 currently. If he cooperates in the current period (with payoff of 1) and defects in the following period (with expected payoff of 1), his total payoff is the same. The arguments for player 1 apply when there are at least two periods remaining.

This inductive argument shows that both players are indifferent between cooperating or breaking down cooperation at any point of the game until the end. Let us see what incentives players have in the end of the game if cooperation has not broken down. In the second to last period player 1 has belief ½. Player 2 cooperates or defects with probability ½ (cooperates if he is behavioral and defects if normal). Player 1 cooperates for sure, which is fine because he is indifferent. In the last period, if cooperation has not broken down, then player 1 defects and player 2 cooperates (b/c he is behavioral).

If one of the players has defected before, we also need to check that both players have incentives to defect in all subsequent periods. The proof of this fact is omitted. It is true for sufficiently low probabilities, as on the equilibrium path above.

Reputation greater than on the balanced path: *If $p_m > 1/2^{(m-1)/2}$ when m is odd, then player both players cooperate and player 2's reputation does not change in this period. If $2/2^{m/2} > p_m > 1/2^{m/2}$ when m is even then player 1 cooperates and player 2 defects with such probability (greater than ½) that his reputation rises to $p_{m-1} = 2/2^{m/2}$, and the players enter the balanced path. If $p_m \geq 2/2^{m/2}$ when m is even then both players cooperate.*

Reputation greater than on the balanced path: *If $p_m < 1/2^{(m-1)/2}$ when m is odd, then player 1 will defect in this period, and player 2 will mix so that conditional on cooperating his reputation rises to $2/2^{(m-1)/2}$. If player 2 happens to defect, then player 1 becomes convinced that player 2 is normal and both players defect forever. If player 2 happens to cooperate, then player 2 defects in the following period (because the tit-for-tat player is supposed to defect) and player 1 will mix between cooperating and defecting. If player 1 happens to defect, then the players defect forever. If player 1 happens to cooperate, then we enter the balanced path with case A. If $p_m < 1/2^{m/2}$ when m is even, then the entry into the balanced path is more complicated than in the previous case, but it can be constructed analogously.*

<u>Problem 2.</u>

Consider a Cournot duopoly, in which two players simultaneously choose quantities q_1 and q_2, sell at price $12-q_1-q_2$, so that the payoffs of firms 1 and 2 are $q_1(12-q_1-q_2)$ and $q_2(12-q_1-q_2)$ respectively.

(a) Find the unique pure strategy Nash equilibrium.

The best response functions are $q_1 = (12-q_2)/2$ and $q_2 = (12-q_1)/2$, so $q_1=q_2=4$ in a unique Nash equilibrium.

(b) Find the monopoly quantity, which maximizes $Q(12 - Q)$.

$Q = 6$.

Now, consider a 2-period game, in which players play a Cournot duopoly in period 1 and a coordination game shown in the figure below in period 2:

quantities q_1 and q_2
price $12 - q_1 - q_2$
profit: $q_1(12 - q_1 - q_2)$ to firm 1 and
$q_2(12 - q_1 - q_2)$ to firm 2

	C	D
C	1, 1	0, 0
D	0, 0	2, 2

For this problem, please focus exclusively on pure strategy SPE of this game.

(c) Show that there is no pure strategy SPE, in which both firms are producing half the monopoly quantity in period 1.

Suppose there was an SPE with both firms producing 3 in period 1. Then each firm can gain 4.5(12-4.5-3) -3(12-3) = 20.25 – 18 = 2.25 in period 1 by deviating to quantity 4.5. Yet, in period 2 a deviation can cause a loss of at most 1 (because the stage game in the second period has two pure strategy Nash equilibria with payoffs (1,1) and (2,2)). Therefore, the immediate gain from a deviation is greater than the loss in future payoff.

(d) Find the best SPE, in which both firms choose the same quantity in period 1? Hint: Firms want to produce less, closer to splitting the monopoly quantity, but they are constrained by rewards and punishments available in the second period to provide them with incentives against deviations. Find the smallest pair of quantities that can be enforced by the available rewards and punishments in the second period.

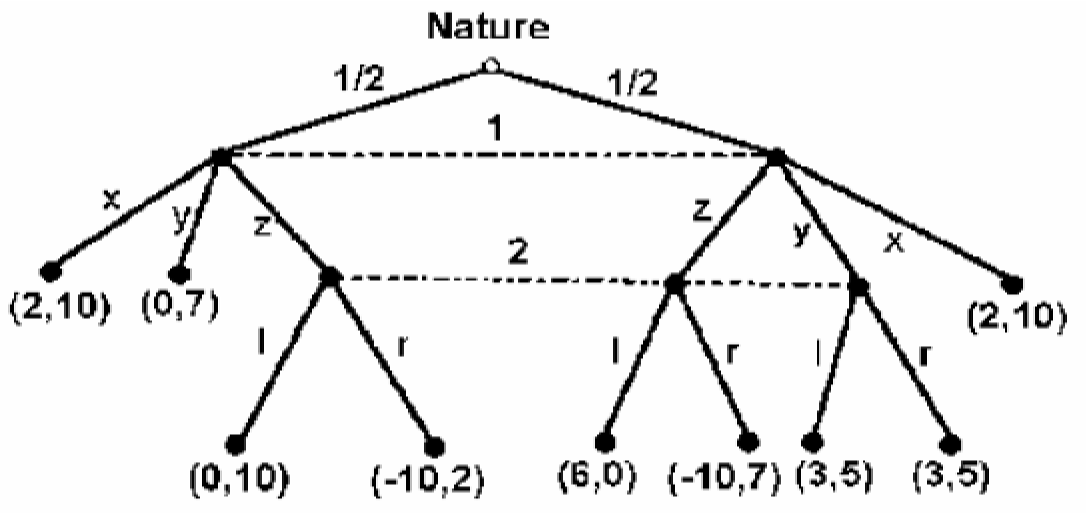
Consider a quantity pair (q, q) with $q \in [0. 6]$. The gain from the most profitable deviation from this quantity pair is

$$(12-q)^2/4 - q(12-2q) = 36 - 6q + q^2/4 - 12q + 2q^2 = 36 - 18q + 9q^2/4 = (6 - 3/2\, q)^2$$

Quantity pair (q, q) can be enforced in period 1 if and only if $(6 - 3/2q)^2 \leq 1 \Leftrightarrow q \in [10/3, 14/3]$, i.e. the gain in the first period is less than or equal to the maximal loss of payoff in period 2. On this range of quantities, q = 10/3 is the best in terms of first-period payoff. Therefore, the following SPE maximizes achieves maximal payoff:

On the equilibrium path firms are supposed to choose quantities (10/3, 10/3) in period 1 and play a Nash equilibrium with payoffs (2,2) in the second period. If one of the firms deviates in period 1, they play a Nash equilibrium with payoffs (1,1) in period 2.

<u>Problem 3</u>. Consider the following two player game in extensive form:



A. Characterize the set of Perfect Bayesian Equilibria.

B. Characterize the set of Sequential Equilibria. Hint: What does Sequential Equilibrium imply about player 2's beliefs in his information set?

(a) Notation: a strategy for 1 in this game is given by (p_1, p_2) where p_1, p_2 and $(1 - p_1 - p_2)$ are the probabilities 1 places on x, y, and z respectively. A strategy for 2 is given by q, the probability 2 places on choosing l. Beliefs for 1 will always be $(\frac{1}{2}, \frac{1}{2})$. Beliefs for 2 are $(\mu_1, \mu_2, 1 - \mu_1 - \mu_2)$.

Observe first that y is strictly dominated for 1, so in any NE, and hence in any PBE, we have $p_2 = 0$.

We look first for eqa where 1 places positive probability on z, so that 2's information set is reached with positive probability. In this case, Bayes' law implies 2's beliefs must be $\mu_1 = \mu_2 = \frac{1}{2}$. Now we can pin down 2's strategy: $\pi_2(l) = 5 > 4.5 = \pi_2(r)$, so 2 must be playing l. This pins downs 1's strategy: $\pi_1(x) = 2 < 3 = \pi_1(z)$, so 1 must play z. Thus, the unique PBE where 1 places positive probability on z is

$$\text{1's strategy: } z \quad \text{2's strategy: } l \quad \text{2's beliefs: } \mu_1 = \mu_2 = \frac{1}{2} \qquad (13)$$

Now look for eqa where 1 plays x only. For 1 to prefer x over z, we must have restrictions on 2's strategy q:

$$\begin{aligned} \pi_1(x) \geq \pi_1(z) \quad &\Rightarrow 2 \geq \tfrac{1}{2} \cdot q \cdot 0 + \tfrac{1}{2} \cdot q \cdot 6 + \tfrac{1}{2}(1-q)(-10) + \tfrac{1}{2}(1-q)(-10) \\ &\Rightarrow 2 \geq 13q - 10 \\ &\Rightarrow q \leq \tfrac{12}{13}. \end{aligned}$$

Now 2's strategy gives us restrictions on 2's beliefs.

If q $\in (0, \frac{12}{13}]$, then 2 must be indifferent between l and r. Thus, we have

$$\pi_2(l) = \pi_2(r) \quad \Rightarrow \mu_1 \cdot 10 + \mu_2 \cdot 0 + (1 - \mu_1 - \mu_2)5 = \mu_1 \cdot 2 + \mu_2 \cdot 7 + (1 - \mu_1 - \mu_2)5$$
$$\Rightarrow 8\mu_1 = 7\mu_2.$$

Thus, we have PBE:

1's strategy: x
2's strategy: q where $q \in (0, \frac{12}{13}]$
2's beliefs: (μ_1, μ_2) where $8\mu_1 = 7\mu_2$ and $\mu_1 + \mu_2 \leq 1$.

If q $= 0$, then 2 must (weakly) prefer r to l. Thus, we have

$$\pi_2(l) \leq \pi_2(r) \quad \Rightarrow \mu_1 \cdot 10 + \mu_2 \cdot 0 + (1 - \mu_1 - \mu_2)5 \leq \mu_1 \cdot 2 + \mu_2 \cdot 7 + (1 - \mu_1 - \mu_2)5$$
$$\Rightarrow 8\mu_1 \leq 7\mu_2.$$

Thus, we have PBE:

1's strategy: x
2's strategy: r
2's beliefs: (μ_1, μ_2) where $8\mu_1 \leq 7\mu_2$ and $\mu_1 + \mu_2 \leq 1$.

(b) First note that any PBE in which all information sets are reached with positive probability is also a sequential equilibrium, so the eqm given in (13) is a SE.

Now consider the eqa where 1 plays x only. Observe that for any strictly mixed strategy for 1, the beliefs generated for 2 must have $\mu_1 = \mu_2$. It follows that the limit of beliefs generated by any sequence of strictly mixed strategies must also have $\mu_1 = \mu_2$. Therefore, in any sequential equilibrium (SE) we must have $\mu_1 = \mu_2$. Combining this with the previous restrictions on μ_1 and μ_2, the set of candidate SE's is reduced to

1's strategy: x
2's strategy: q where $q \in [0, \frac{12}{13}]$
2's beliefs: $\mu_1 = \mu_2 = 0$.

Can we in fact find strictly mixed strategies and corresponding beliefs that converge to the equilibrium? Yes. We just need to pick the strategies carefully, making sure the weight 1 places on z goes to zero faster than the weight on y.

Let $\sigma_1^{(n)} = (1 - \varepsilon^n - \varepsilon^{2n}, \varepsilon^n, \varepsilon^{2n})$. Let $\sigma_2^{(n)} = q + \varepsilon^n$. Observe that as $n \to \infty$, the strategies converge to the equilibrium strategies. Also,

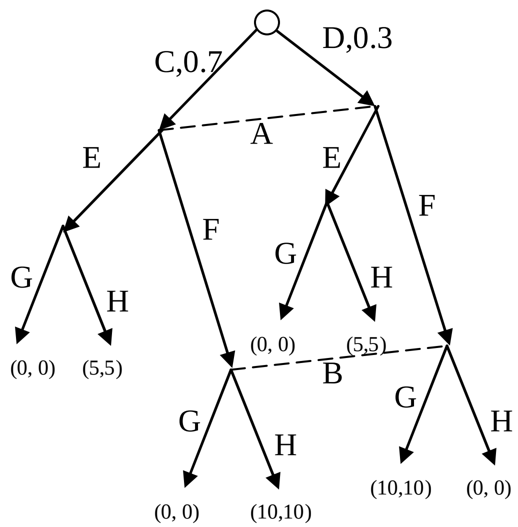
$$\mu_1^{(n)} = \mu_2^{(n)} = \frac{\frac{1}{2}\varepsilon^{2n}}{\frac{1}{2}\varepsilon^{2n} + \frac{1}{2}\varepsilon^{2n} + \frac{1}{2}\varepsilon^n} = \frac{\varepsilon^n}{2\varepsilon^n + 1} \to 0 \quad \text{and}$$

$$\mu_3^{(n)} = \frac{\frac{1}{2}\varepsilon^n}{\frac{1}{2}\varepsilon^{2n} + \frac{1}{2}\varepsilon^{2n} + \frac{1}{2}\varepsilon^n} = \frac{1}{2\varepsilon^n + 1} \to 1.$$

The beliefs converge to the eqm beliefs, so the candidates are indeed SE.

Problem 4. Consider the following game played by two parties, A and B. First, "nature" chooses either C or D. C is chosen with probability 0.7, and D is chosen with probability 0.3. Second, party A chooses either E or F. Party A does not observe nature's choice when it makes this choice. Next, party B chooses either G or H. Prior to making this choice, party B observes the choice of party A; party B also observes nature's choice if A has chosen E, but does not observe nature's choice if A has chosen F. Payoffs are determined as follows: A and B always receive the same payoff; the payoff is 0 if A chooses E and B chooses G, regardless of nature's choice; the payoff is 5 if A chooses E and B chooses H, regardless of nature's choice; the payoff is 0 if nature chooses C, A chooses F, and B chooses G; the payoff is 10 if nature chooses C, A chooses F, and B chooses H; the payoff is 10 if nature chooses D, A chooses F, and B chooses G; and the payoff is 0 if nature chooses D, A chooses F, and B chooses H.

A. Draw the extensive form of this game.



B. Please write a list of all strategies for each player.

Player A: E, F
Player B: GGG, GGH, GHG, GHH, HGG, HGH, HHG, HHH, where the first letter is the action that follows CE, second letter is the action that follows F, and third letter is the action that follows DE.

C. Identify all pure strategy Subgame Perfect Equilibria for this game.

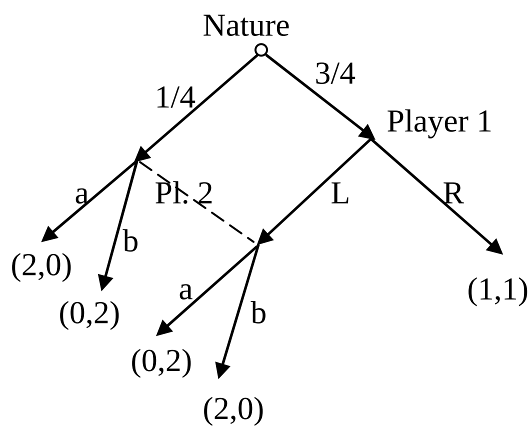
The game has three subgames. By looking at the small subgames, we conclude that player B's strategy has to be HGH or HHH in any SPE. If player B plays HGH in the entire game, player A must play E. Because HGH is also a best response of player B in the entire game, it follows that (E, HGH) is a SPE. Similarly, if player B plays HHH, then player A must play F. We can easily check that (F, HHH) is a SPE also. We conclude that there are two SPEs: (E, HGH) and (F, HHH).

D. Identify all pure strategy Perfect Bayesian Equilibria for this game.

Strategies in any PBE must constitute a SPE. Therefore, we need to find all PBE in which (E, HGH) or (F, HHH) are played. For (E, HGH) to be a PBE, player 2 must place a probability of at least ½ on the right node of his information set. For (F, HHH) *to be a PBP, player 2's beliefs must come from Bayes rule in his information set, so he must put probability 0.3 on the right node.*

Remark. *There is no Sequential equilibrium with strategies (E, HGH).*

<u>Problem 6.</u> In the game below, please find all Perfect Bayesian Equilibria.



To find all PBE, we consider three cases of what can happen in a PBE.

<u>Case 1</u>: Suppose player 1 chooses R for sure. Then player 2 must put probability 1 on the upper left node by Bayes rule. Given those beliefs, player 2 must choose b. Given 2's action, player 1 must choose L, a contradiction.

<u>Case 2</u>: Suppose player 1 chooses L for sure. Then player 2 must put probability ¼ on the upper left node by Bayes rule. Given those beliefs, player 2 must choose a. Given 2's action, player 1 must choose R, a contradiction.

<u>Case 3</u>. Suppose player 1 mixes with strictly positive probabilities. Then in order for player 1 to be indifferent between L and R, player 2 must also mix, placing equal probabilities on a and b. This strategy of player 2 is optimal only if he puts equal probability on each node in his information set. Because those beliefs must satisfy the Bayes rule, player 1 must choose L with probability 1/3.

We obtained a PBE only in Case 3: (1/3L + 1/3R, ½a +½b, 2's beliefs (½, ½)).

<u>Problem 7.</u>

(a) Consider a monopolist with marginal cost c = 0 or 3 who faces a demand of 12 – Q. The monopolist's profit is given by (12 – Q – c) Q. Find the monopoly quantity and profit.

The monopoly quantity is $Q = (12-c)/2$*, and profit is* $(12-c)^2/4$.

(b) Consider a two firms, who play Cournot duopoly. Firm 1 has marginal cost 0 and firm 2 has marginal cost c. They face a demand of $12 – q_1 – q_2$ and the profits are given by $q_1(12 – q_1 – q_2)$ and $q_2(12 – q_1 – q_2 - c)$. Find the Nash equilibrium (as a function of c, assuming that c = 0 or 3).

The best response function of firm 1 is given by $q_1 = (12-q_2)/2$. The best response function of firm 2 is $q_2 = (12 – c – q_1)/2$. Solving a system of two simultaneous equations, we find that $q_1 = 4 + c/3$ and $q_2 = 4 – 2c/3$.

(c) Suppose that in period 1 firm 1 watches the behavior of firm 2, which has cost c = 0 or 3 and faces demand 12 – Q. Firm 1 does not know firm 2's cost, but can see the quantity chosen by firm 2 in period 1. Between periods 1 and 2, firm 2 decides whether to enter or not. Entering involves a fixed cost of 20. If firm 1 does not enter, firm 2 gets to be a monopolist in period 2. If firm 2 enters, then firms 1 with marginal cost 0 and firm 2 with the same marginal cost as in the first period play Cournot duopoly, facing demand $12 – q_1 – q_2$. Find all separating equilibria.

In a separating equilibrium the type of firm 2 is revealed in period 1. If firm 1 enters, it gets Cournot profit against firm 2 with cost c, which is $(4 + c/3)^2 = 25$ and 16 for c = 0 and 3 respectively. Comparing entry profit with entry cost, we find that firm 1 will enter if and only if firm 2 is the high-cost type.

By the standard argument, the high-cost firm 2 will choose its preferred quantity Q_3 = 4.5 in period 1. After the entrant enters, the quantities chosen are (5, 2) in period 2.

Let us find the range of quantities Q_0 of the low-cost firm 2 in period 1, for which there exists a separating equilibrium. If we take any separating equilibrium with a quantity pair (Q_0, Q_3) and change beliefs to c=3 everywhere off the equilibrium path then we get a separating equilibrium with the same first-period quantity pair. In this separating equilibrium the high-cost firm might be tempted to imitate the low-cost firm. If it does so, it will choose quantity Q_0 in period 1 and get profit of $(9 – Q_0)Q_0$ instead of 4.5^2. In period 2 there is no entry and firm 2 gets a profit of 4.5^2 instead of $(12-8+c/3-c)(4-2c/3)=4$. Such a deviation is not profitable if $(9-Q_0)Q_0 \le 4 \Leftrightarrow$

$$Q_0 \le \frac{9-\sqrt{65}}{2} \quad \text{or} \quad Q_0 \ge \frac{9+\sqrt{65}}{2}$$

The most tempting deviation of the low-cost firm is to choose its preferred quantity 6 and be perceived a high-cost type. If it deviates, the entrant will enter and choose quantity q_1=5 in period 2. The best response to quantity 5 is 3.5. Such a deviation is not profitable if

$$(12-Q_0)Q_0+36 \ge 36+3.5(12-5-3.5) \Leftrightarrow (12-Q_0)Q_0 \ge 12.25 \Leftrightarrow Q_0 \in \left[\frac{12-\sqrt{95}}{2}, \frac{12+\sqrt{95}}{2}\right]$$

We conclude that a separating equilibrium exists for $Q_0 \in \left[\frac{9+\sqrt{65}}{2}, \frac{12+\sqrt{95}}{2}\right]$. One can specify a range of off-equilibrium path beliefs for any pair of quantities (Q_0, Q_3=4.5) with $Q_0 \in \left[\frac{9+\sqrt{65}}{2}, \frac{12+\sqrt{95}}{2}\right]$, but beliefs that assign probability 1 to a high-cost incumbent work.