Econ 201A. Part II.
Professor Yuliy Sannnikov

Solutions to Problem Set 3

Problem 1. Consider the symmetric auction environment discussed in class. There are N bidders with valuations uniformly distributed on the interval $[0, \alpha]$. Consider an all-pay auction: the highest bidder wins, and all bidders pay their bid. Please find the equilibrium bidding function. Please compute the seller’s expected revenue and compare it with that from the first price auction and a second price auction.

Solution. Denote the equilibrium bidding function by σ. Then, when a bidder submits bid b, he wins the auction with probability

$$F(\sigma^{-1}(b))^{N-1}.$$

The bidder’s expected payoff is

$$\pi(v, b) = v\, F(\sigma^{-1}(b))^{N-1} - b$$

This expression has to be maximized when $b = \sigma(b)$. Taking the first order condition with respect to b, we get

$$v\, (N-1)\sigma^{-1}(b)'f(\sigma^{-1}(b))F(\sigma^{-1}(b))^{N-2} - 1=0$$

Substituting $b = \sigma(v)$ and $\sigma^{-1}(b)' = 1/\sigma'(v)$, we get

$$\sigma'(v) = v(N-1)\, f(v)F(v)^{N-2} \Rightarrow \sigma(v) = \int_0^v \frac{(N-1)\, v^{N-1}}{\alpha^{N-1}} = \frac{N-1}{N}\frac{v^N}{\alpha^{N-1}}$$

The seller’s expected revenue is

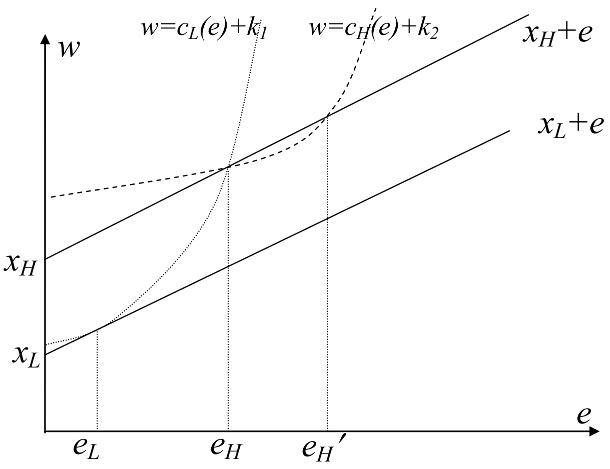
$$N\int_0^\alpha \frac{N-1}{N}\frac{v^N}{\alpha^{N-1}} f(v)dv = \frac{N-1}{N+1}\frac{v^{N+1}}{\alpha^N}\bigg|_0^\alpha = \frac{N-1}{N+1}\alpha,$$

the same as in the first and second price auctions.

Problem 2: (Similar to Kreps, Problem 2 in Chapter 17). Consider the following signaling environment from class: 2 types that have cost of education $c_L(e)$ and $c_H(e)$ respectively with $c_L'>c_H'>0$, $c_L''>0$ and $c_H''>0$. The productivities of the two types are x_L+e and x_H+e. Assume that $c_L'(0) < 1$ and $c_H'(0) < 1$, but $c_L'(e)\to\infty$ and $c_H'(e)\to\infty$ as $e\to\infty$.

(a) Are there separating equilibria where one type (or both) chooses more than one level of education? If they exist, do they satisfy the intuitive criterion? By a separating equilibrium we mean that if one type chooses a given education level with positive probability, then the other type does not.

Solution. Such a separating equilibrium can exist, but it always fails the intuitive criterion. The picture below shows a possible separating equilibrium, in which type L chooses the efficient level e_L and type H chooses levels e_H and e_H' with positive probability. In this example the market always believes it is type L whenever it sees an off-equilibrium education level.



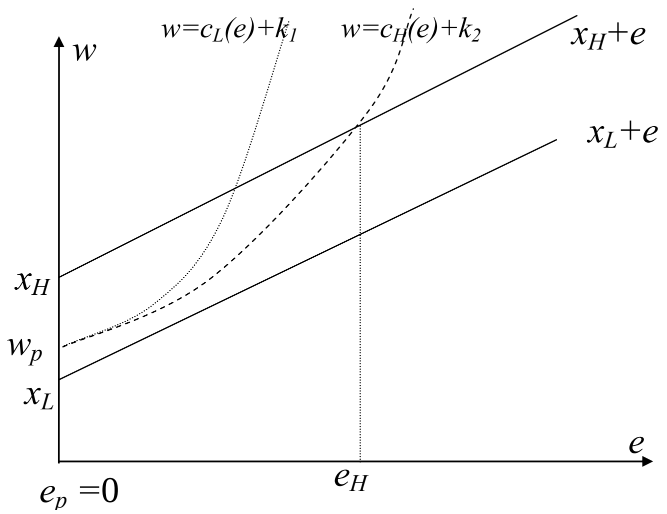
To prove that any such equilibrium fails the intuitive criterion, note that the low type must always choose only one education level – the efficient one. The high type chooses to levels of education, e_H and e_H' with positive probability. Because indifference curves are concave and increasing, the low type strictly prefers his education level to any level between e_H and e_H', even if he is inferred to be the high type. Therefore, by the intuitive criterion the market must believe that it is the high type whenever it sees any education level between e_H and e_H'. Given that, the high type will deviate.

(b) Is there any pooling equilibrium where both types choose more than one level of education? By a pooling equilibrium we mean that every education level chosen by one type with positive probability is chosen by the other type with positive probability.

Solution. No. Suppose such equilibrium existed, with education and wage levels (e_1, w_1) and (e_2, w_2). Since both types must choose these education levels with positive probabilities, they must be indifferent between choosing e_1 and e_2. However, it is impossible for two indifference curves for a high and a low type to cross twice at points (e_1, w_1) and (e_2, w_2), because $c_L' > c_H'$.

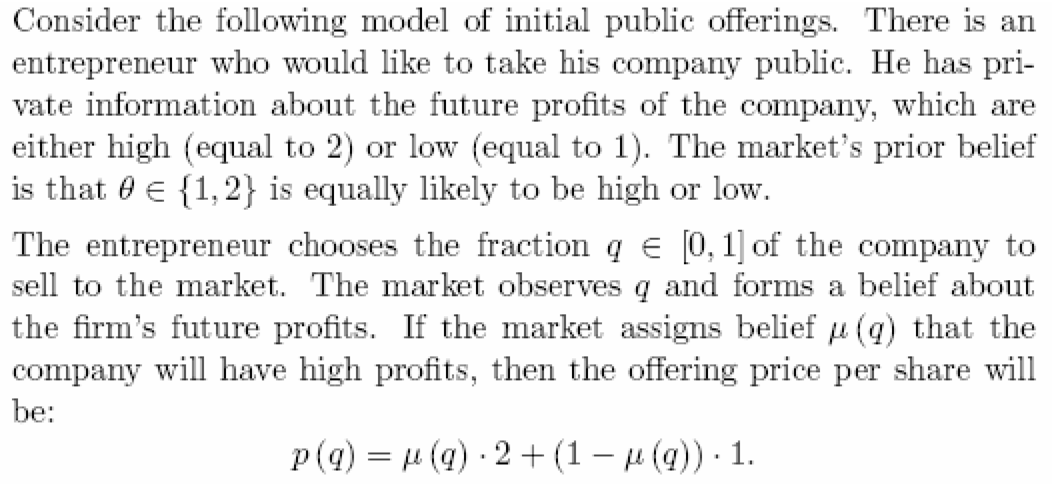
(c) A hybrid equilibrium is one in which some education levels are chosen by one type only and others are chosen by both types. Are there any hybrid equilibria?

Solution. Yes, please see the picture below:



This picture illustrates a hybrid equilibrium, in which the low type always chooses education level 0 and the high type mixes between 0 and e_H.

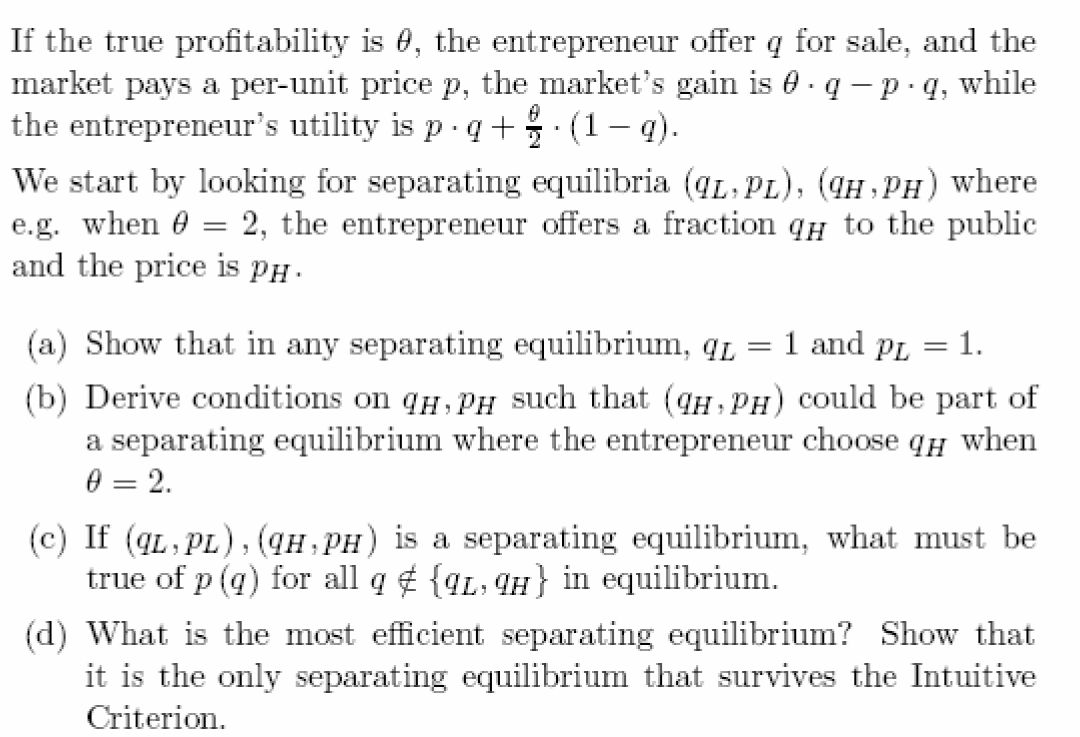
Problem 3.



Consider the following model of initial public offerings. There is an entrepreneur who would like to take his company public. He has private information about the future profits of the company, which are either high (equal to 2) or low (equal to 1). The market's prior belief is that $\theta \in \{1, 2\}$ is equally likely to be high or low.

The entrepreneur chooses the fraction $q \in [0, 1]$ of the company to sell to the market. The market observes q and forms a belief about the firm's future profits. If the market assigns belief $\mu(q)$ that the company will have high profits, then the offering price per share will be:

$$p(q) = \mu(q) \cdot 2 + (1 - \mu(q)) \cdot 1.$$



If the true profitability is θ, the entrepreneur offer q for sale, and the market pays a per-unit price p, the market's gain is $\theta \cdot q - p \cdot q$, while the entrepreneur's utility is $p \cdot q + \frac{\theta}{2} \cdot (1 - q)$.

We start by looking for separating equilibria (q_L, p_L), (q_H, p_H) where e.g. when $\theta = 2$, the entrepreneur offers a fraction q_H to the public and the price is p_H.

(a) Show that in any separating equilibrium, $q_L = 1$ and $p_L = 1$.

(b) Derive conditions on q_H, p_H such that (q_H, p_H) could be part of a separating equilibrium where the entrepreneur choose q_H when $\theta = 2$.

(c) If $(q_L, p_L), (q_H, p_H)$ is a separating equilibrium, what must be true of $p(q)$ for all $q \notin \{q_L, q_H\}$ in equilibrium.

(d) What is the most efficient separating equilibrium? Show that it is the only separating equilibrium that survives the Intuitive Criterion.

Solution.

(a) In a separating equilibrium, the market must believe that $\theta = 1$ when it observes q_L, so $p_L = 1$. It is impossible that $q_L < 1$, because otherwise the low type would deviate to $q = 1$ and improve his payoff.

(b) $p_H = 2$ comes from market inferences in a separating equilibrium. Two conditions must hold

$2\ q_H + (1-q_H) \geq 1$ (type H does not deviate) – always holds

$2\ q_H + ½\ (1-q_H) \leq 1$ (type L does not deviate) – holds if $q_H \leq 1/3$

(c) We need to make sure that neither type would want to deviate off-equilibrium. For $q < q_H$ price $p(q)$ could be any number between 1 and 2. For $q \in (q_H, 1)$, the following two conditions must hold:

$2\ q_H + (1-q_H) \geq\ p(q)\ q + (1-q)$ (type H does not deviate) $\Leftrightarrow p(q) \leq q_H/q + 1$

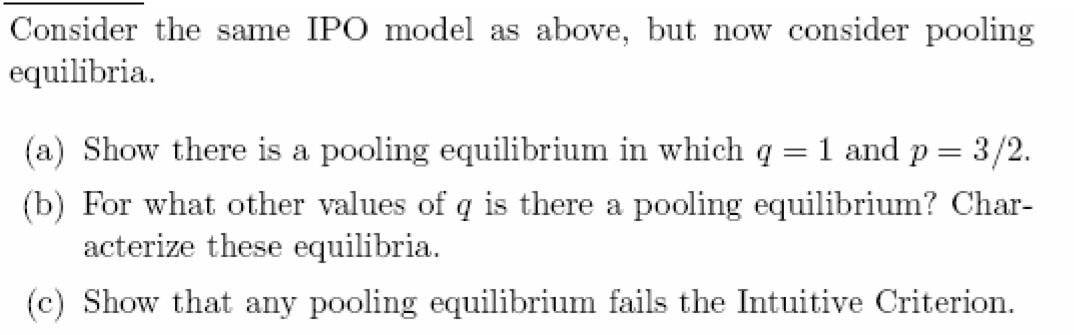
$1 \geq\ p(q)\ q + ½\ (1-q)$ (type L does not deviate) $\Leftrightarrow p(q) \leq 1/(2q) + 1/2$

We must have $p(q) \leq \min(q_H/q + 1, 1/(2q) + ½)$.

(d) In the most efficient separating equilibrium the high type sells the most shares, i.e. $q_H = 1/3$. In this equilibrium the intuitive criterion does not place any restrictions on beliefs: both types would like to deviate to any $q \in (1/3, 1)$ if the market inferred them to he the high type, and neither type would like to deviate to $q < 1/3$ even if the market believed it was the high type.

We also need to show that any other equilibrium fails the intuitive criterion. Suppose $q_H < 1/3$. Then the low type would not want to deviate to $q \in (q_H, 1/3)$ even if $p(q) = 2$. Therefore, the intuitive criterion implies that the market believes a deviator to be the high type if it observes $q \in (q_H, 1/3)$. Given those beliefs, the high type would deviate into this range.

Problem 4.



Consider the same IPO model as above, but now consider pooling equilibria.

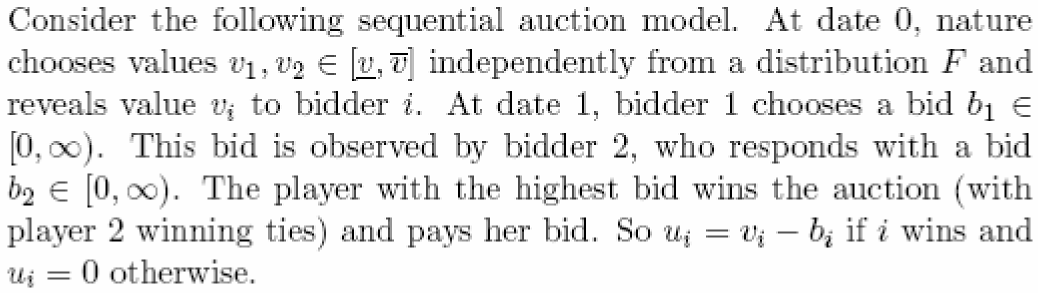
(a) Show there is a pooling equilibrium in which $q = 1$ and $p = 3/2$.

(b) For what other values of q is there a pooling equilibrium? Characterize these equilibria.

(c) Show that any pooling equilibrium fails the Intuitive Criterion.

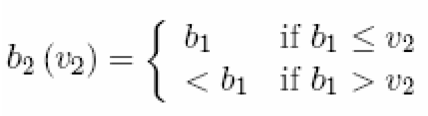
Solutions. (a) Price $p = 3/2$ is justified by the fact that both types are equally likely. If the market believes the type to be low for $q < 1$, then neither type would want to deviate because payoffs are increasing in q and decreasing in p.
(b) The price in any pooling equilibrium is $p = 3/2$. The main condition to ensure is that the low type would not want to deviate to $q = 1$, i.e. $3/2\ q_P + ½\ (1-q_P) \geq 1 \Leftrightarrow q_p \geq ½$. Any such q_p can arise in a pooling equilibrium if the market believes that $\theta = 1$ for off-equilibrium values of q.
(c) Even if given price 2, type 1 would not want to deviate to any $q < \frac{2}{3} q_p$. Type 2 would strictly want to deviate to $q > \frac{1}{2} q_p$ if given price 2. Therefore, by the intuitive criterion the market must believe the type to be $\theta = 2$ anywhere in the range $\left(\frac{1}{2} q_p, \frac{2}{3} q_p\right)$. This gives a profitable deviation to type 2, so all pooling equilibria fail the intuitive criterion.

Problem 5.



Consider the following sequential auction model. At date 0, nature chooses values $v_1, v_2 \in [\underline{v}, \overline{v}]$ independently from a distribution F and reveals value v_i to bidder i. At date 1, bidder 1 chooses a bid $b_1 \in [0, \infty)$. This bid is observed by bidder 2, who responds with a bid $b_2 \in [0, \infty)$. The player with the highest bid wins the auction (with player 2 winning ties) and pays her bid. So $u_i = v_i - b_i$ if i wins and $u_i = 0$ otherwise.

(A) Show that in any perfect Bayesian equilibrium, bidder two must use the strategy.



$$b_2(v_2) = \begin{cases} b_1 & \text{if } b_1 \leq v_2 \\ < b_1 & \text{if } b_1 > v_2 \end{cases}$$

Solution. The payoff of player 2 is independent of his beliefs about player 1. If player 1 chooses $b_1 < v_2$, then player 2 can obtain a positive payoff by winning the auction. He makes the smallest bid that allows him to win. If $b_1 > v_2$ then player 2 would always obtain a negative payoff if he wins. It is best to place a bid less than b_1 and let player 1 win.

(B) Use your answer to (A) to solve for the perfect Bayesian equilibrium of the auction.

Solution. For the uniform distribution: If player 1 bids b_1 he wins the auction in case player 2's valuation falls in the range $[\underline{v}, b_1]$ with probability $\frac{b_1 - \underline{v}}{\overline{v} - \underline{v}}$. Player 1's expected payoff is $\frac{b_1 - \underline{v}}{\overline{v} - \underline{v}}(v_1 - b_1),$ which is maximized when $b_1 = (\underline{v} + v_1)/2$.

(C) Compute the seller's expected revenue and compare it to that in the standard first price and second price auctions.

Solution. For the uniform distribution: The seller always receives the bid that player 1 made, which is $\underline{v} + (\overline{v} - \underline{v})/4$ in expectation. In a standard first and second price auction, the seller's revenue would be larger. It would be $\underline{v} + (\overline{v} - \underline{v})/3,$ as computed in class.

Problem 6.

Consider the following version of the Prisoners' Dilemma with asymmetric payoffs:

	C	D
C	3, 2	-1, 3
D	5, -1	0, 0

This game is repeated infinitely with discount factor δ.

A. What are the payoffs in the worst possible subgame perfect equilibrium? Prove that it is impossible to achieve payoffs worse than those.

B. What is the lowest discount factor δ^* for which the players achieve cooperation in a SPE?

C. Is it possible to achieve payoffs better than (0, 0) in a SPE for discount factors lower than δ^*? Hint: Think about a SPE in which players alternate between (C, D) and (D, C) on the equilibrium path.

Solution. (a) The payoffs in the worst SPE are (0, 0). They are achieved by the repetition of Nash. The reason why it is impossible to achieve payoffs worse than 0 for either player is that each player can guarantee himself a payoff of at least 0 by playing D in every period.

(b) We are interested in the SPE that has (C, C) in every period on the equilibrium path. We need to check that neither player would want to deviate once from (C, C). According to part (a) the worst available punishment has payoff 0, so **both** of the following conditions need to hold:

$$5 + 0 \leq 3/(1-\delta) \Leftrightarrow \delta \geq 2/5 \text{ (deviation of player 1)}$$
$$3 + 0 \leq 2/(1-\delta) \Leftrightarrow \delta \geq 1/3 \text{ (deviation of player 2)}$$

We find that $\delta^* = 2/5$.

(c) Yes. Consider a SPE in which player alternate between regimes (C, D) and (D, C) as long as neither player deviates, and deviations are punished by Nash equilibrium forever. We need to check that single deviations are not profitable:

$$3 + 0 \leq 5 - \delta + 5\delta^2 - \delta^3 \ldots = (5 - \delta)/(1-\delta^2) \text{, always holds (player 1 in regime (D, C))}$$
$$0 \leq -1 + 3\delta + 5\delta^2 - \delta^3 \ldots = (-1 + 3\delta)/(1-\delta^2) \text{, holds if } \delta \geq 1/3 \text{ (player 2 in regime (D, C))}$$
$$0 \leq -1 + 5\delta - \delta^2 + 5\delta^3 \ldots = (-1 + 5\delta)/(1-\delta^2) \text{, holds if } \delta \geq 1/5 \text{ (player 1 in regime (C, D))}$$
$$2 \leq 3 - \delta + 3\delta^2 - \delta^3 \ldots = (3 - \delta)/(1-\delta^2) \text{, always holds (player 2 in regime (C, D))}$$

This shows that alternating between (C, D) and (D, C) can happen on the equilibrium path if $\delta \geq 1/3$ (note that $1/3 < \delta^*$). If players start in regime (C, D), they get strictly positive payoffs of

$$(-1 + 5\delta)/(1-\delta^2) \qquad \text{and } (3 - \delta)/(1-\delta^2)$$

Problem 7.

(Similar to Kreps, Problem 9, Chapter 17). In a particular population everyone runs the risk of losing \$1000 randomly. Each person's chance to lose \$1000 depends on the individual: fraction x of the population loses \$1000 with probability .1 while the other fraction $1-x$ loses \$1000 with probability 0.6. Individuals know their types, but insurance companies do not. Each individual is risk averse and has the same von Neumann-Morgenstern utility function given by $u(x) = -e^{-\lambda x}$ with $\lambda > 0$, but the insurance companies are risk neutral. Each individual decides whether to seek insurance or not. If he chooses to seek insurance, he approaches a number of insurance companies who simultaneously name premiums for full insurance P. The individual accepts the lowest premium. He pays the insurance company P. The company covers the \$1000 loss in the event it happens.

A. What expected utility does an individual of each type gets if he chooses to go without insurance? What utility does he get if he chooses to take insurance?

> *Answer*. Expected utility without insurance is $-qe^{1000\lambda} - (1-q)$, where q is the probability of losing \$1000. Expected utility with insurance is $-e^{P\lambda}$.

B. Please think intuitively about whether the less risky individuals get insurance or not. How does the answer depend on λ and x (i.e. is there a separating equilibrium for large λ or small λ, large x or small x)? Why?

> *Answer.* The less risky individual definitely gets insurance if λ is sufficiently large (i.e. he is very risk averse, so he would be willing to accept insurance even

if on significantly unfair terms). Also it is easier for the low-risk type to get insurance if x is large because the premium in a pooling equilibrium would be close to fair for the low-risk type. However, even if x is arbitrarily large there may be a separating equilibrium where the premium is tailored to the high-risk type.

C. For what range of parameters x and λ is there a separating equilibrium? For what range of parameters is there a pooling equilibrium? Please verify your conjecture from part B.

Solution. In a separating equilibrium only the high-risk types get insured and the premium is competitive for the high-risk type, i.e. P = 0.6 × \$1000 = \$600. There is a separating equilibrium if the low-risk type prefers to go uninsured, i.e.

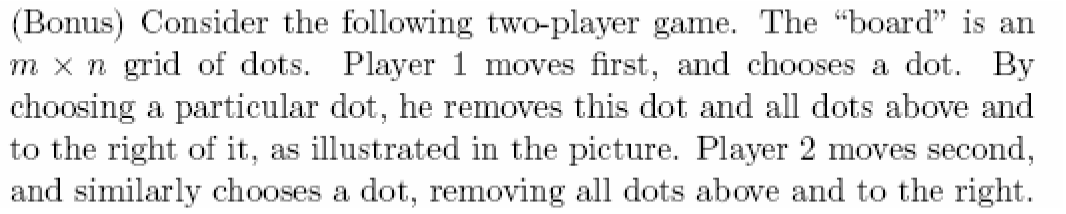
$$-0.1\, e^{1000\lambda} - 0.9 \geq - e^{600\lambda} \Leftrightarrow \lambda < 0.00568,$$

i.e. if the low-risk type is not too risk-averse. This confirms our first guess in part B: if the low-risk type is sufficiently risk-averse, he would be willing to accept insurance at unfair terms. A pooling equilibrium exists if the low-risk type prefers to take insurance with premium P = 100 x + 600 (1-x), i.e.

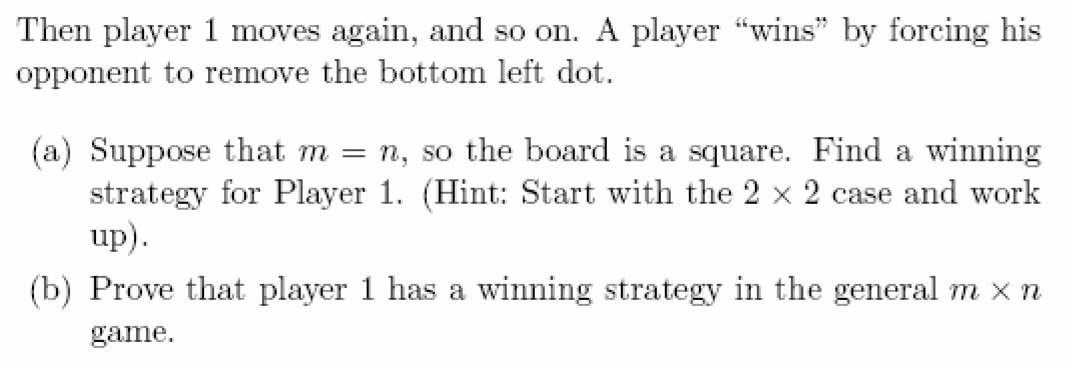
$$-0.1\, e^{1000\lambda} - 0.9 \leq - e^{(600-500x)\lambda} \Leftrightarrow \lambda > f(x),$$

where $f(x) : [0, 1] \rightarrow [0, \infty)$ is an decreasing function of x with $f(0) = 0.00568$ and $f(1) = 0$. This confirms our second guess in part B: as x increases, the range where the pooling equilibrium exists becomes larger, but the range where the separating equilibrium exist does not change.

The Last Problem:



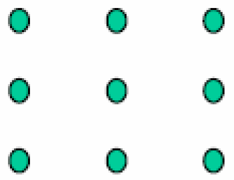
(Bonus) Consider the following two-player game. The "board" is an $m \times n$ grid of dots. Player 1 moves first, and chooses a dot. By choosing a particular dot, he removes this dot and all dots above and to the right of it, as illustrated in the picture. Player 2 moves second, and similarly chooses a dot, removing all dots above and to the right.



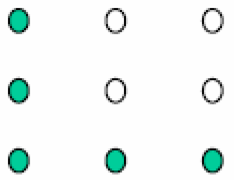
Then player 1 moves again, and so on. A player "wins" by forcing his opponent to remove the bottom left dot.

(a) Suppose that $m = n$, so the board is a square. Find a winning strategy for Player 1. (Hint: Start with the 2×2 case and work up).

(b) Prove that player 1 has a winning strategy in the general $m \times n$ game.



Initial 3x3 Board



Board after Center
Square is removed.

Solution.

(a) Player 1 removes the upper right $n-1 \times n-1$ square with his first move, which creates an L-shaped configuration of dots. After that whenever player 2 removes k upper dots, player 1 will respond by removing the same number of right-side dots. Whenever player 2 removes k right-side dots, player 1 will respond by removing the same number of upper dots. As a result, the L-shaped configuration is always symmetric after player 1's move. In the end of the game player 2 will be forced to remove the last dot.

(b) Consider all configurations of dots that could possibly arise in the course of the play. Let's call a configuration of dots *winning* if the player who moves from it has a winning strategy, and *losing* if his opponent has a winning strategy. Winning and losing configurations can be characterized by the following properties: Any move from a losing configuration leads to a winning configuration for the opponent. For any winning configuration, there is a move that leads to a losing configuration.

Let's prove that an $n \times m$ arrangement of dots is a winning configuration, i.e. the first player has a winning strategy. Suppose on the contrary that the $n \times m$ arrangement of dots is a losing configuration. Then any configuration that can be obtained in one move is a winning configuration. In particular, the configuration with the upper right dot removed is a winning configuration. Then there has to be a move from this configuration that leads to a losing configuration. Whatever this move is, this losing configuration will look like the original $n \times m$ rectangle with some upper right rectangle removed. However, this configuration can be obtained from the original rectangle in one move! This is a contradiction, since we assumed that the original rectangle is a losing configuration. Therefore, the first player who moves from an $n \times m$ rectangle has a winning strategy.