Econ 201a Part II
Fall 2004

Extra problems

Problem 1.

Consider a finite game, and suppose that it is solvable by iterative elimination of strictly dominated strategies. Prove that it has a unique Nash equilibrium and that this coincides with the iterated dominance solution.

Solution.
1. First, Nash equilibrium survives iterative elimination of strictly dominated strategies (IESDS).

Assume to the contrary that there exists a NE that is eliminated. This means that there is at least one strategy component of the NE that is ruled out by this process. Without loss of generality, consider the first strategy component that is eliminated. Let this be player j's strategy s′(j). Then it follows that there exists a strategy s″(j) for player j that strictly dominates s″(j) for the strategies in the Nash equilibrium. This however contradicts that s′(j) had to be a best response to the rest of the Nash equilibrium strategy profile.

Note now that if we could assume that NE had always existed we would be done. Unless you prove this, the proof so far is incomplete.

The nice thing about this problem is that you can prove it without the general existence property of NE, only from the fact that the game is solvable by IESDS. So…

2. Unique (!) IESDS surviving strategy profile is NE[1]*.*

Suppose that the surviving strategy profile (s(1),s(2),…s(j),…s(N)) is not a NE. Then at least for one player there is a profitable deviation. I.e. there is an eliminated strategy, say for player j, such that s″(j) gives higher payoff to player j than his surviving strategy, conditional on the fact that the other players play their surviving strategies. This however cannot be the case.

Consider the round when s″(j) was eliminated. Since all strategies in (s(1),s(2),…s(j),…s(N)) must have survived up to this round it follows that (s(1),s(2),…s″(j),…s(N) cannot give a higher payoff to player j than (s(1),s(2),…s(j),…s(N)), otherwise s(j) could not have survived the round when s″(j) got eliminated.

Finally, since Nash survives IESDS and IESDS gives a unique prediction for the game the NE must be unique as well.

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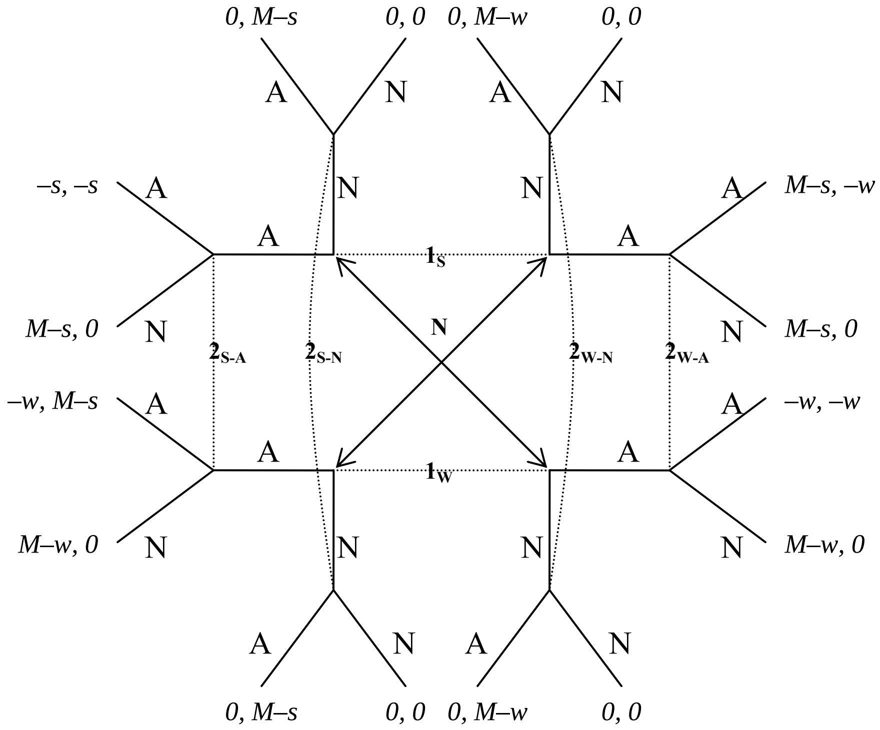
[1] Note that in the general case IESDS might well give a bigger set of strategies as a prediction of the game than the NE.

Problem 2.

Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack." In addition, each army is either "strong" or "weak" with equal probability (the draws for each army are independent), and an army's type is known only to its general. General 1 makes a decision whether to attack first, general 2 observes that decision, and then decides. Payoffs are as follows: The island is worth *M* if captured. An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is *s* if it is strong and *w* if it is weak, where *s* < *w* <*M/2*.

(a) Draw the game tree.
(b) Find all PBE of this game.

Solution.



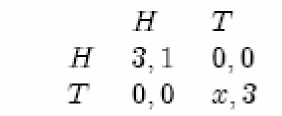
A. *If the first general attacks, the second general will not attack if he is weak, because a weak second general would have no chance of capturing the island. Then both types of generals attack in period 1. Indeed, by not attacking they get a payoff of 0. By attacking, they capture the island with probability at least 1/2 and get a payoff of at least M/2 - w > 0 or M/2 - s > 0 depending on their type. Then, if the second general sees attack, he must believe that the first general is strong or* *weak with equal probabilities. Given those beliefs, the second general will attack if he is strong, because then his expected payoff is M/2 – s > 0.*

To finish the description of PBE we must describe what happens if the first general does not attack. Then both types of general 2 will attack. Their beliefs about the type of general 1 can be arbitrary in a PBE because the information set where the first general does not attack is off the equilibrium path.

Remark: *In a sequential equilibrium these beliefs can also be arbitrary, but both types of general 2 must have the same belief.*

<u>Problem 3</u>.

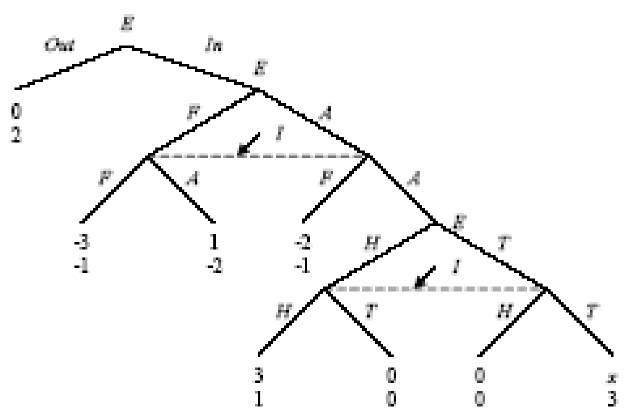
Two firms, E (for entrant) and I (for incumbent) play the following game. First, E chooses "In" or "Out". If E chooses "Out" then payoffs are (0, 2) (the convention here is that E's payoff is given first). If E chooses "In", then E and I play a simultaneous move game where each chooses to either "fight" (F) or "accommodate" (A). (F,F) yields payoffs of (−3, −1), (F,A) yields payoffs of (1, −2), and (A,F) yields payoffs of (−2, −1). In the event that the firms choose (A,A) at this stage, the game continues. Each firm simultaneously chooses either "here" (H) or "there" (T). Payoffs after these choices are given as follows:



	H	T
H	$3, 1$	$0, 0$
T	$0, 0$	$x, 3$

A. Draw the game tree.
B. Find all subgame perfect equilibria of this game when *x* = 1. Please be sure to consider mixed strategies.
C. Find all SPE when *x*=¾.
D. Find all SPE when *x* = ½.

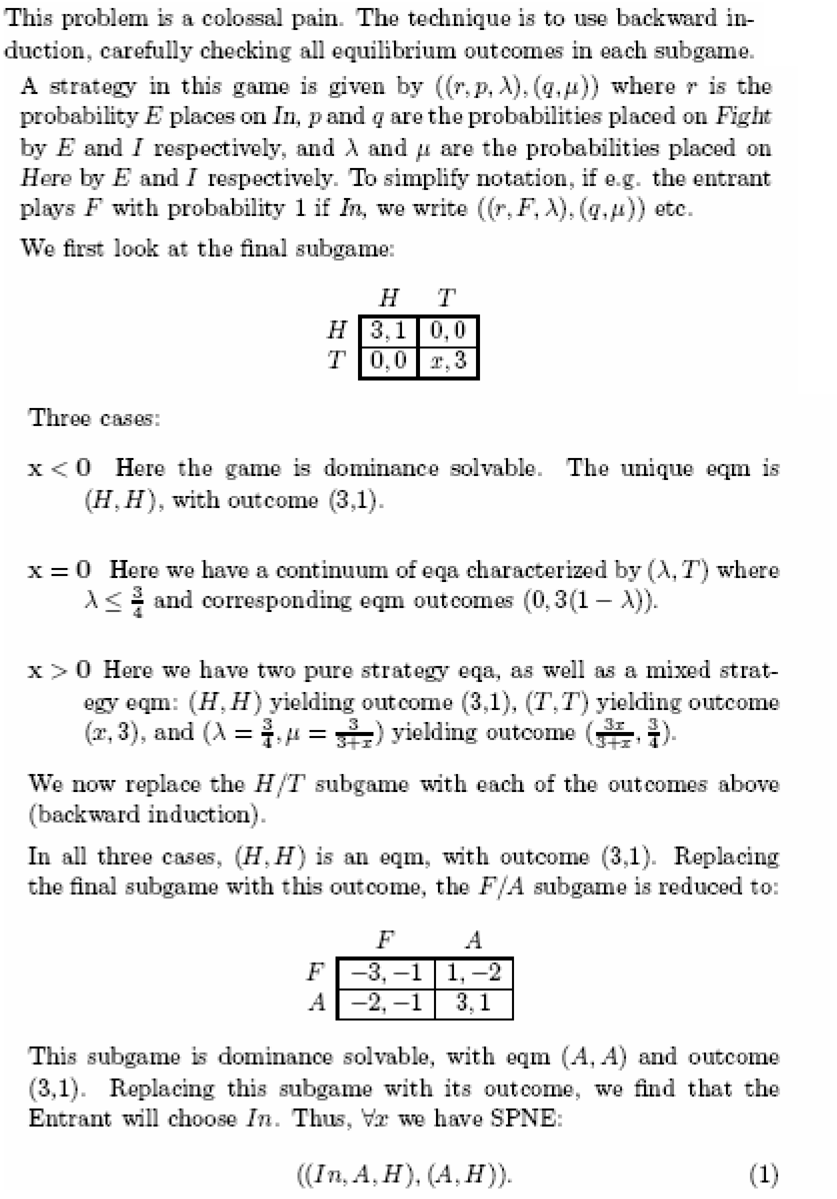
Solution.



A.

The following solution derives SPE for all values of *x*.

B.



This problem is a colossal pain. The technique is to use backward induction, carefully checking all equilibrium outcomes in each subgame.

A strategy in this game is given by $((r, p, \lambda), (q, \mu))$ where r is the probability E places on *In*, p and q are the probabilities placed on *Fight* by E and I respectively, and λ and μ are the probabilities placed on *Here* by E and I respectively. To simplify notation, if e.g. the entrant plays F with probability 1 if *In*, we write $((r, F, \lambda), (q, \mu))$ etc.

We first look at the final subgame:

	H	T
H	$3, 1$	$0, 0$
T	$0, 0$	$x, 3$

Three cases:

x < 0 Here the game is dominance solvable. The unique eqm is (H, H), with outcome (3,1).

x = 0 Here we have a continuum of eqa characterized by (λ, T) where $\lambda \leq \frac{3}{4}$ and corresponding eqm outcomes $(0, 3(1-\lambda))$.

x > 0 Here we have two pure strategy eqa, as well as a mixed strategy eqm: (H, H) yielding outcome (3,1), (T, T) yielding outcome $(x, 3)$, and $(\lambda = \frac{3}{4}, \mu = \frac{3}{3+x})$ yielding outcome $(\frac{3x}{3+x}, \frac{3}{4})$.

We now replace the H/T subgame with each of the outcomes above (backward induction).

In all three cases, (H, H) is an eqm, with outcome (3,1). Replacing the final subgame with this outcome, the F/A subgame is reduced to:

	F	A
F	$-3, -1$	$1, -2$
A	$-2, -1$	$3, 1$

This subgame is dominance solvable, with eqm (A, A) and outcome (3,1). Replacing this subgame with its outcome, we find that the Entrant will choose In. Thus, $\forall x$ we have SPNE:

$$((In, A, H), (A, H)). \tag{1}$$

Case x = 0 The F/A subgame becomes

	F	A
F	$-3, -1$	$1, -2$
A	$-2, -1$	$0, 3(1-\lambda)$

This game has a unique eqm – a mixed strategy eqm given by ($p = \frac{4-3\lambda}{5-3\lambda}, q = \frac{1}{2}$) with outcome $(-1, -1)$. It follows that E will play Out. Thus, when $x = 0$ we have SPNE

$$\left(\left(Out, \frac{4-3\lambda}{5-3\lambda}, \lambda\right), \left(\frac{1}{2}, T\right)\right) \quad \text{for any } \lambda \leq \frac{3}{4}. \tag{2}$$

Case x > 0, outcome = (x,3) Now the F/A subgame becomes

	F	A
F	$-3, -1$	$1, -2$
A	$-2, -1$	$x, 3$

Since x is in the normal form, we have subcases.

(a) **x > 1** Here the F/A subgame is dominance solvable. The eqm is (A, A), with outcome $(x, 3)$. It follows that E will play In. Thus, when $x > 1$, we have SPNE

$$((In, A, T), (A, T)). \tag{3}$$

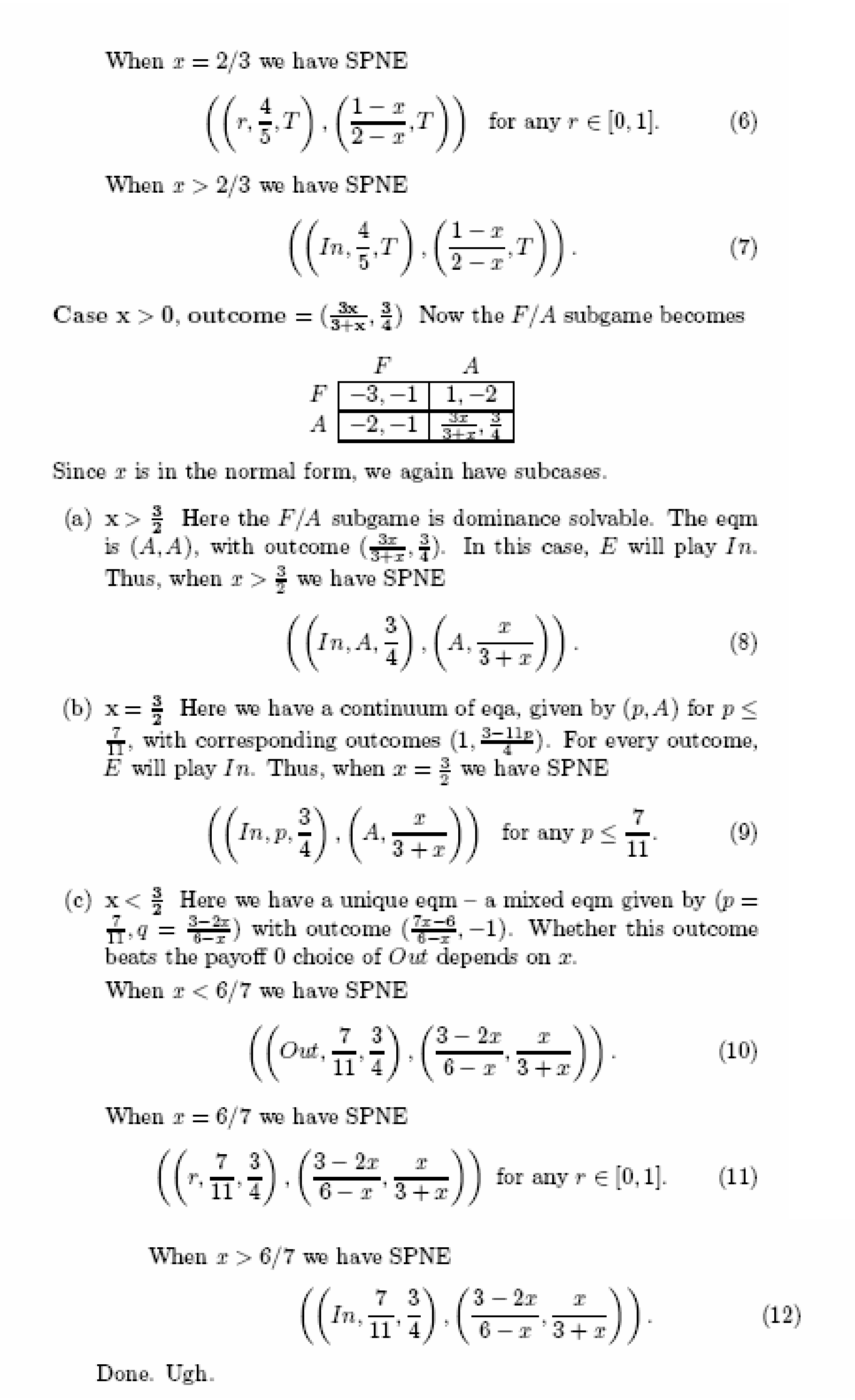
(b) **x = 1** Here we have a continuum of eqa given by (p, A) where $p \leq \frac{4}{5}$, and corresponding outcomes $(1, 3-5p)$. For any outcome, E will play In. Thus, when $x = 1$, we have SPNE

$$((In, p, T), (A, T)) \quad \text{for any } p \leq \frac{4}{5}. \tag{4}$$

(c) **x < 1** Here we have a unique eqm – a mixed eqm given by ($p = \frac{4}{5}, q = \frac{1-x}{2-x}$) with outcome ($\frac{3x-2}{2-x}, -1$). Whether this outcome beats the payoff 0 choice of Out depends on x.

When $x < 2/3$ we have SPNE

$$\left(\left(Out, \frac{4}{5}, T\right), \left(\frac{1-x}{2-x}, T\right)\right). \tag{5}$$



When $x = 2/3$ we have SPNE

$$\left(\left(r, \frac{4}{5}, T\right), \left(\frac{1-x}{2-x}, T\right)\right) \quad \text{for any } r \in [0,1]. \tag{6}$$

When $x > 2/3$ we have SPNE

$$\left(\left(In, \frac{4}{5}, T\right), \left(\frac{1-x}{2-x}, T\right)\right). \tag{7}$$

Case x > 0, outcome = $(\frac{3x}{3+x}, \frac{3}{4})$ Now the F/A subgame becomes

	F	A
F	$-3, -1$	$1, -2$
A	$-2, -1$	$\frac{3x}{3+x}, \frac{3}{4}$

Since x is in the normal form, we again have subcases.

(a) x > $\frac{3}{2}$ Here the F/A subgame is dominance solvable. The eqm is (A, A), with outcome $(\frac{3x}{3+x}, \frac{3}{4})$. In this case, E will play In. Thus, when $x > \frac{3}{2}$ we have SPNE

$$\left(\left(In, A, \frac{3}{4}\right), \left(A, \frac{x}{3+x}\right)\right). \tag{8}$$

(b) x = $\frac{3}{2}$ Here we have a continuum of eqa, given by (p, A) for $p \leq \frac{7}{11}$, with corresponding outcomes $(1, \frac{3-11p}{4})$. For every outcome, E will play In. Thus, when $x = \frac{3}{2}$ we have SPNE

$$\left(\left(In, p, \frac{3}{4}\right), \left(A, \frac{x}{3+x}\right)\right) \quad \text{for any } p \leq \frac{7}{11}. \tag{9}$$

(c) x < $\frac{3}{2}$ Here we have a unique eqm – a mixed eqm given by $(p = \frac{7}{11}, q = \frac{3-2x}{6-x})$ with outcome $(\frac{7x-6}{6-x}, -1)$. Whether this outcome beats the payoff 0 choice of Out depends on x.

When $x < 6/7$ we have SPNE

$$\left(\left(Out, \frac{7}{11}, \frac{3}{4}\right), \left(\frac{3-2x}{6-x}, \frac{x}{3+x}\right)\right). \tag{10}$$

When $x = 6/7$ we have SPNE

$$\left(\left(r, \frac{7}{11}, \frac{3}{4}\right), \left(\frac{3-2x}{6-x}, \frac{x}{3+x}\right)\right) \quad \text{for any } r \in [0,1]. \tag{11}$$

When $x > 6/7$ we have SPNE

$$\left(\left(In, \frac{7}{11}, \frac{3}{4}\right), \left(\frac{3-2x}{6-x}, \frac{x}{3+x}\right)\right). \tag{12}$$

Done. Ugh.

To summarize:

(b) p=1: Continuum of SPE

- (*In*, A, H) (A, H)
- (*In*, p, T) (A, T) 0≤p≤4/5 (many people found only p=0 and/or p=4/5.)
- (*In*, 7/11, 3/4) (1/5, ¼)

(c) p=¾: 3 SPE

(*In,* A, H) (A, H)
(*In*, 4/5, T) (1/5, T)
(*Out*, 7/11, ¾) (2/7, 1/5)

(d) p= ½: 3 SPE
(*In*, A, H) (A, H)
(*Out*, 4/5, T) (1/3, T)
(*Out*, 7/11, ¾) (4/11, 1/7)

Problem 4.

Two firms, A and B, are in a market that is declining in size. The game starts in period 0, and the firms can compete in periods 0,1,2,3,... (i.e. indefinitely) if they so choose. Duopoly profits in period t for firm A are equal to 101-10t and they are 10-t for firm B. Monopoly profits (which are earned if a firm is the only one left in the market) are 500-25t for firm A and 50-2t for firm B.

Suppose that at the start of each period the two firms each must decide to "stay in" or to "exit" if it is still active (they do so simultaneously if both are still active). Once a firm exits, it is out of the market forever and earns zero in each period thereafter. Firms maximize the (undiscounted) sum of their profits. Characterize the subgame perfect equilibria for this game.

Solution.

Recall that in an extensive form game, a (pure) strategy consists of an action taken at each information set. In this game, information sets consist of monopoly nodes and duopoly nodes. That is, in the extensive form, each player has exactly $t+1$ time t information sets - one duopoly node (no one exited in time $0 \ldots t-1$) and t monopoly nodes (opponent exited at each time $0 \ldots t-1$).

Using backward induction on the subgames that begin with monopoly nodes, actions in monopoly nodes are easily specified:

Firm A:
$t = 1 \ldots 19 \Rightarrow$ stay in
$t = 20 \Rightarrow \alpha$ stay in, $1-\alpha$ out
$t > 20 \Rightarrow$ out

Firm B:
$t = 1 \ldots 24 \Rightarrow$ stay in
$t = 25 \Rightarrow \beta$ stay in, $1-\beta$ out
$t > 25 \Rightarrow$ out

We now must specify actions taken at duopoly nodes. Again we rely on backward induction.

In the duopoly subgame beginning at any $t > 25$, for each firm, any strategy specifying "out" at time t strictly dominates any strategy specifying "stay in". Thus, "out" is chosen by each firm at each duopoly node with $t > 25$.

Consider the $t = 25$ duopoly subgame. For firm A, any strategy specifying "out" strictly dominates any strategy specifying "in", so A plays out at $t = 25$. Firm B mixes between "out" and "in".

Consider the $t = 21 - 24$ duopoly subgames. Here A's dominant strategy is to exit immediately regardless. It follows that B stays in and earns monopoly profits.

Consider the $t = 20$ duopoly subgame. Here we see that it is a dominant strategy for B to play "in", because in the continuation subgame, A will drop out in the next period and B will reap 4 periods of monopoly profit. Thus, A must play "out".

Consider the $t = 11-19$ duopoly subgames. For each t, it is a dominant strategy for B to play "in" because (by backward induction) in the continuation subgame, A will drop out in the next period. Thus A plays "out".

Consider the $t = 10$ duopoly subgame. "in" is a dominant strategy for A because duopoly profits are positive. "in" is also a dominant strategy for B because as before, A will drop out in the next period.

In the $t = 0 - 9$ subgames, duopoly profits are positive for both, so both play "in".

Thus, the SPE take the following form:

A's strategy:

Monopoly node: Stay in 0-19. Any mixture at 20. Out after 20.
Duopoly node: Stay in 0-10. Exit 11 or later.

B's strategy:

Monopoly or duopoly node: Stay in 0-24. Any mixture at 25. Out 26 or later.