Economics 201A
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This class is premised on the belief that a somewhat informal intuitive explanation with many examples is much more useful than a formal mathematical dry explanation. Ultimately, in order to be able to analyze games, one needs to develop a feel for the tricks that are useful to argue about games. Even though formality instills a (false) feeling of confidence, ultimately it stiffens us up, and takes away our flexibility to take leaps forward. I believe that any result must first be developed informally through leaps of loose logic, and then made rigorous by finding appropriate formalism to justify it.

Let me give an example how formalism can restrict our imagination, and how a loose informal explanation can convey a lot more. Informal words can convey the same meaning as a dry formal explanation, except it takes less time and effort to understand. Here is a *formal and rigorous* definition of a simultaneous-move game with private information from Osborne and Rubinstein:

Definition 25.1 A **Bayesian game** consists of
- a finite set N (the set of **players**)
- a finite set Ω (the set of **states**)

and for each player $i \in N$
- a set A_i (the set of **actions** available to player i)
- a finite set Y_i (the set of **signals** that may be observed by player i) and a function $y_i : \Omega \rightarrow Y_i$ (the signal function of player i)
- a probability measure p_i on Ω (the **prior belief** of player i) for which $p_i(\tau_i^{-1}(y_i)) > 0$ for all $y_i \in Y_i$
- a preference relation $\geq_i$ on the set of probability measures over $A \times \Omega$ (the **preference relation** of player i), where $A = \times_{j \in N} A_j$.

To me, this definition is extremely hard to understand and most importantly, hard to work with. It makes a game look like a bunch of formal sets and functions. Most importantly, this definition does not have a flavor of a game. Let me present a nearly equivalent informal definition:

Definition. N players participate in the game. First, the nature chooses the state of fundamentals ω from the probability distribution p. Each player privately sees a random signal about fundamentals $y_i \in Y_i$ with probability $q_i(y_i \mid \omega)$. After seeing his signal, each player privately takes an action $a_i \in A_i$. The payoff of player i is given by $g_i(a, \omega)$, where $a = (a_1, \ldots a_N)$ is the vector of actions that the players took.

These definitions are nearly equivalent, but the second definition conveys a lot more flavor and color about the game. From this definition one can imagine each player, the information that he gets, and the agony about reaching the decision about which action to take. Let me also give an example of such a game (from the introductory lecture):

Example: (Auction) A seller holds a first-price auction for an object. There are N bidders, each of whom has valuation v_i for the object. Valuations are distributed independently across players uniformly on [0, 1]. Each bidder decides how much to bid. The object goes to the highest bidder, who pays his bid and gets the payoff of

$$v_i - p_i$$

All the other bidders get a payoff of 0.

To fit this example with the definition above, the state of fundamentals is the vector of valuations (v_1, … v_N), and the signal of bidder i is her own valuation. The strategy of a player in this game is a function from her signals to actions, e.g. $b : [0, 1] \to [0, \infty)$ in the example of an auction. A (Bayesian) Nash equilibrium is a profile of strategies, such each player's strategy maximizes her expected payoff given the strategies of other players.

Even though formalism stiffens our mind, nevertheless sometimes we feel the necessity to have a formal foundation to have some ground to stand on (and some you may desire to see a formal exposition). Therefore, let me formalize the concepts I have gone through in this class so far.

<u>Normal-form games (aka simultaneous-move games).</u>

Definition. A **normal form game** consists of

- a finite set {1,2,… N} of players
- for each player i ∈ N a nonempty set A_i of actions
- for each player i ∈ N a payoff function $g_i : A_1 \times A_2 ... \times A_N \to \Re$

Definition. A pure strategy Nash equilibrium of a normal form game is a profile of actions $(a_1^*, a_2^*, ..., a_N^*) \in A_1 \times A_2 ... \times A_N$ such that for every player *i*

$$g_i(a_i^*, a_{-i}^*) \geq g_i(a_i, a_{-i}^*) \quad \text{for all } a_i \in A_i$$

Definition. A mixed strategy of player i is a probability distribution over a player's actions. The set of mixed strategies is denoted by $\Delta(A_i)$.

Definition. A **mixed extension** of the normal form game (N, (A_i), (g_i)) is the game (N, ($\Delta(A_i)$), (g_i)) in which $g_i : (\Delta A_1) \times (\Delta A_2) ... \times (\Delta A_N) \to \Re$ assigns to each $\alpha : (\Delta A_1) \times (\Delta A_2) ... \times (\Delta A_N)$ the expected value of $g_i(a)$ with respect to the probability measure α over $A_1 \times A_2 ... \times A_N$.

Definition. A **mixed strategy Nash equilibrium** of a normal form game is a Nash equilibrium of its mixed extension.

Proposition. (Will not be proved in class) *Every finite strategic game has a mixed strategy Nash equilibrium.*

Proposition. *Let G = (N, (A_i), (g_i)) be a finite strategic game. Then $\alpha^* : (\Delta A_1) \times (\Delta A_2) ... \times (\Delta A_N)$ is a mixed strategy Nash equilibrium of G if and only if for every player i ∈N every pure strategy in the support of α_i^* is a best response to α_{-i}^*.*

Note: the support of a probability distribution is the set of all elements, to which it assigns a positive probability.

Here is an example for a completely formal proof:

Formal Proof. First, let us show that if α^* is a Nash equilibrium, then any action in the support of α_i^* is a best response to α_{-i}^*. Suppose not: there is an action a_i in the support of α_i^* that is not a best response to α_{-i}^*. Then by linearity of g_i in α_i player i can increase his payoff by transferring probability from a_i to an action that *is* a best response; hence α_i^* is not a best response to α_{-i}^*.

Second, let us show that if every action in the support of α_i^* is a best response to α_{-i}^*, then α_i^* is a best response to α_{-i}^*. Suppose not. Then there is a mixed strategy α_i' that gives a higher expected payoff than does α_i^* in response to α_{-i}^*. Then again by the linearity of g_i in α_i, α_i' must give a higher payoff than some action in the support of α_i^* in response to α_{-i}^*, so that not all actions in the support of α_i^* are best responses to α_{-i}^*.
QED

Extensive-form games

Here is a typical formal definition of an extensive-form game:

Definition. An **extensive game with perfect information** has the following components:

- A set N (the set of players)
- A set H of histories of actions that satisfies the following three properties
  - $\varnothing \in H$
  - If $a^1a^2 \ldots a^K \in H$ (where K may be infinite) then any subhistory $a^1a^2 \ldots a^L$ with $L < K$ is also in H
  - If any subhistory of an infinite history $a^1a^2 \ldots$ is in H, then the infinite history itself is in H

  (The set of terminal histories, i.e. those that are not subhistories of any other history, is denoted by Z).
- A function $P : H \backslash Z \to N$.
- A payoff function $g_i : Z \to R$ for each player $i \in N$.

Remark: Games with simultaneous moves, e.g. repeated Prisoners' Dilemma or Durable Goods Monopoly, do not fit this definition.

Example: Rubinstein bargaining game. Nonterminal histories take the form

$\{x_1,\text{reject},x_2,\text{reject}, \ldots x_n, \text{reject}\}$ (if n is not the last period)
and $\{x_1,\text{reject},x_2,\text{reject}, \ldots x_n\}$

Terminal histories take the form

$\{x_1$,reject,x_2,reject, … accept$\}$
and $\{x_1$,reject,x_2,reject, … x_n, reject$\}$ (if n is the last period)

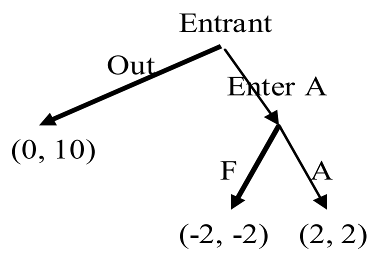
Defintion: A **strategy** of player $i \in N$ in an extensive game with perfect information $(N, H, P, (g_i))$ is a function that assigns an action $A(h)$ to each nonterminal history $h \in H \setminus Z$ for which $P(h) = i$.

Denote by $O(s)$ the terminal history that results when players follow strategy profile s.

Definition: A **Nash equilibrium of an extensive game with perfect information** $(N, H, P, (g_i))$ is a strategy profile s^* such that for every player $i \in N$ we have

$$g_i(O(s_i^*, s_{-i}^*)) \geq g_i(O(s_i, s_{-i}^*)) \quad \text{for all alternative strategies } s_i \text{ of player } i$$

Example: Nash equilibrium (which is not subgame perfect):

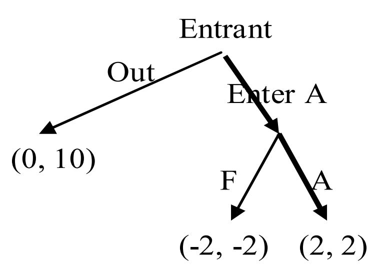


Subgame perfect equilibrium (need to define what is a subgame first):

Definition. The subgame of $\Gamma = (N, H, P, (g_i))$ that follows history $h \in H/Z$ is the extensive game $\Gamma(h) = (N, H|_h, P|_h, (g_i)|_h)$ where $H|_h$ is the set of histories h' for which $(h, h') \in H$, $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$ and $g_i|_h$ is defined by $g_i|_h(h') = g_i(h,h')$.

Definition. A **subgame perfect equilibrium** of an extensive game $\Gamma = (N, H, P, (g_i))$ is a strategy profile s^* which induces a Nash equilibrium in each subgame of Γ.

(This is a somewhat informal definition, but you get the point).
Example: Subgame Perfect Equilibrium:



A good reference that has a formal treatment of game theory is the textbook by Osborne and Rubinstein.