Auctions

A simple bidding problem:

- Seller offers a single indivisible object
- N bidders, $i = 1, \dots, N$
- Each bidder i has a reservation valuation v_i
- Valuations are private information
- v_i is a random variable with CDF F on $[v_l, v_h]$
- F is the same for all bidders (symmetry) e.g. F is uniform on $[0, \alpha]$
- Realizations are independent across bidders

This is a case of independent private valuations

1. Second-price sealed bid auction:

Bidders simultaneously submit sealed bids b_i; highest bid wins, but pays the second highest bid. A winner is chosen from among the high bidders with equal probability in the event of a tie.

Claim: Bid $b_i = v_i$ is at least as good as any other bid no matter what other bids are.

Proof. Consider an alternative bid $b_i' > b_i$.

Case 1: There is another bid better than b_i'. Then neither of the bids wins the auction. They both deliver payoff 0.

Case 2: The 2nd highest bid is less than b_i. Then bidder i wins the auction and pays the same second highest bid in both cases. Both b_i' and b_i deliver the same payoff.

Case 3: The highest bid of other players is between b_i and b_i'. Then bid b_i would lose the auction (payoff 0). Bid b_i' would win the auction and pay more than valuation.

How much revenue does the seller receive?

E[$v_{(2)}$] - expectation of the second highest valuation

To compute it, let us find the CDF of $v_{(2)}$.

Prob(2nd highest bid $\leq x$) = Prob(All bids $\leq x$) +
Prob(1st bid $> x$, all other bids $\leq x$) +
Prob(2nd bid $> x$, all other bids $\leq x$) + …
Prob(Nth bid $> x$, all other bids $\leq x$) =
$F(x)^N + N\,(1-F(x))\,F(x)^{N-1}$.

Density: $Nf(x)F(x)^{N-1} + N(N-1)f(x)F(x)^{N-2} - N^2 f(x)F(x)^{N-1} =$

$$N(N-1)f(x)F(x)^{N-2}(1- F(x))$$

Therefore,

$$E\left[v_{(2)}\right] = \int_{v_\ell}^{v_h} vN(N-1)f(v)\left[F(v)\right]^{N-2}\left[1-F(v)\right]dv$$

For the special case when F is uniform on [0, α],

$$\begin{aligned} E\left[v_{(2)}\right] &= N(N-1)\int_0^{\alpha} v\frac{1}{\alpha}\left(\frac{v}{\alpha}\right)^{N-2}\left[1-\frac{v}{\alpha}\right]dv \\ &= \alpha\left(\frac{N-1}{N+1}\right) \end{aligned}$$

2. First-price sealed bid auction

Bidders simultaneously submit sealed bids b_i; highest bidder wins, and pays her own bid. A winner is chosen from among the high bidders with equal probability in the event of a tie.

Payoffs: $v_i - b_i$ for the winner, and 0 for all other bidders.

Strategies: $\sigma_i(v_i)$, mapping valuations, v_i, to bids, b_i.

Assume that: $\sigma_i = \sigma$ is the same for all players

σ is strictly increasing

The probability of winning upon submitting the bid b:

$$F(\sigma^{-1}(b))^{N-1}.$$

The expected payoff

$$\pi(v, b) = (v - b)\, F(\sigma^{-1}(b))^{N-1}$$

F.O.C w.r.t. b must hold when $b=\sigma(v)$

$$-F(\sigma^{-1}(b))^{N-1}+(v-b)(N\text{-}1)\,\sigma^{-1}(b)'f(\sigma^{-1}(b))F(\sigma^{-1}(b))^{N-2}=0$$

Inverse function theorem $\Rightarrow$ $\sigma^{-1}(b)' = 1/\sigma'(v)$, so

$$(v-b)(N\text{-}1)\, f(v)/\sigma'(v) = F(v) \;\Rightarrow\; \sigma'(v) = \frac{(v - b)(N-1)\, f(v)}{F(v)}$$

Exercise: What is $\sigma(v_l)$

σ can be found from $$\sigma(v) = v_l + \int_{v_l}^{v} \frac{(w - b)(N-1)\, f(w)}{F(w)} dw$$

Exercise: solve for $\sigma(v)$ when F is uniform on $[0, \alpha]$

Expected Revenues (uniform case):

$$
\begin{aligned}
E\left[b_{(1)}\right] &= E\left[v_{(1)}\right]\left(\frac{N-1}{N}\right) \\
&= \left(\frac{N-1}{N}\right)\int_0^{\alpha} vNf(v)\left[F(v)\right]^{N-1} dv \\
&= \left(\frac{N-1}{N}\right)\int_0^{\alpha} vN\left(\frac{1}{\alpha}\right)\left[\frac{v}{\alpha}\right]^{N-1} dv \\
&= \left(\frac{N-1}{N}\right)\alpha\left(\frac{N}{N+1}\right) \\
&= \alpha\left(\frac{N-1}{N+1}\right)
\end{aligned}
$$

- same as second price auction

Revenue Equivalence Theorem:

- Assume N identical bidders, indep. private values
- any auction with the following two properties provides the same expected revenue:
  - o the bidder with the highest valuation wins
  - o the bidder with valuation v_l has exp. payoff 0

General proof: Consider any auction structure and an associated equilibrium $\sigma(v)$. Let $p(b)$ denote the equilibrium probability of winning as a function of the bid, and let $t(b)$ denote the expected payment to the auctioneer, given the bid. The expected payoff to a bidder with valuation v is

$$\Pi(v) = vp(\sigma(v)) - t(\sigma(v))$$

$$\text{F.O.C. } vp'(\sigma(v)) - t'(\sigma(v))=0$$

Therefore

$$\frac{\partial \Pi(v)}{\partial v} = p(\sigma(v)) + \sigma'(v)(vp'(\sigma(v)) - t'(\sigma(v))) = p(\sigma(v)) = F(v)^{N-1}$$

- the expected payoff of each type is $\Pi(v) = \int_{v_l}^{v} F(w)^{N-1} dw$

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Highest valuation has CDF $F(v)^N$

(revenue of seller + expected utility of everybody else)

$$\int_{v_l}^{v_h} wNf(w)F(w)^{N-1} dw$$

Expected (a priori) utility of each player:

$$\int_{v_l}^{v_h} \Pi(v) f(v) dv = \int_{v_l}^{v_h} \left(\int_{v_l}^{v} F(w)^{N-1} dw \right) f(v) dv$$

Expected utility of the seller:

$$\int_{v_l}^{v_H} wNf(w)F(w)^{N-1} dw - N \int_{v_l}^{v_h} \left(\int_{v_l}^{v} F(w)^{N-1} dw \right) f(v) dv$$

QED